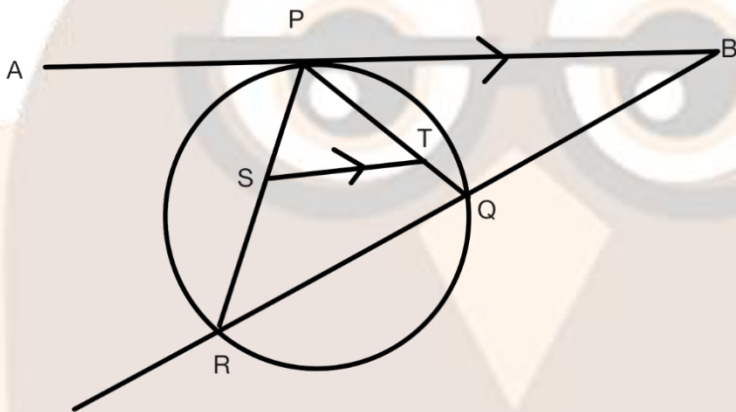
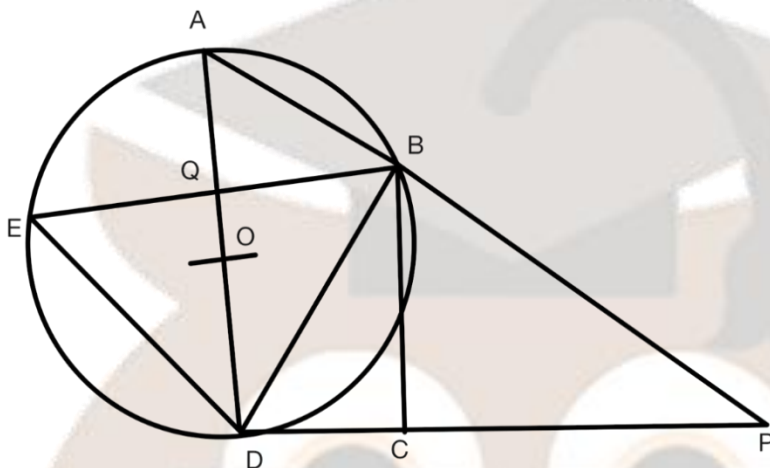


9 ESSENTIAL QUESTIONS TO MASTER

PLANE FIGURES

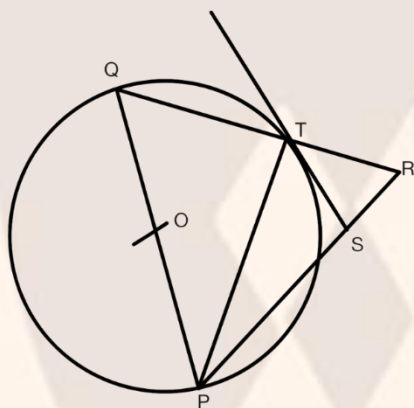
1	A circle passes through points P, Q, R. AB is tangent at P. Q lies on line RB. Points S and T lie on PR and PQ respectively, and $(AB \parallel ST)$. (a) Prove that $(\triangle PST)$ is similar to $(\triangle PQR)$. (b) Hence show that $(PQ \times PT = PR \times PS)$. (c) Determine whether quadrilateral $(STQR)$ is cyclic. 	
2	The diagram shows a circle with centre O. Points A, B, D and E lie on the circle. The line AD is a diameter of the circle, and DCP is the tangent to the circle at D. The lines EB and AD intersect at point Q. The chord AB is produced beyond B to meet the tangent (DCP) at P. It is given that $\angle CBP = \angle DEB$.	



- (a) Show that $\angle BCD = 90^\circ$
 (b) Show that $DP^2 = BP^2 + DA \times BC$

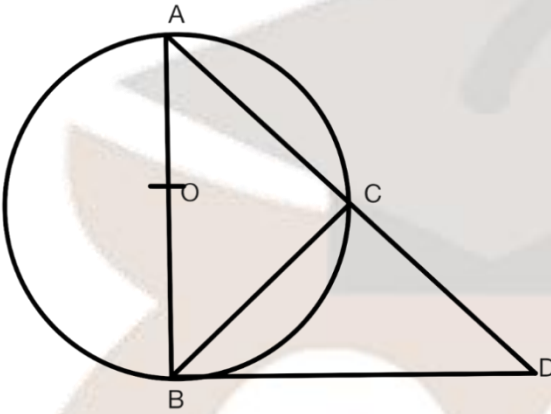
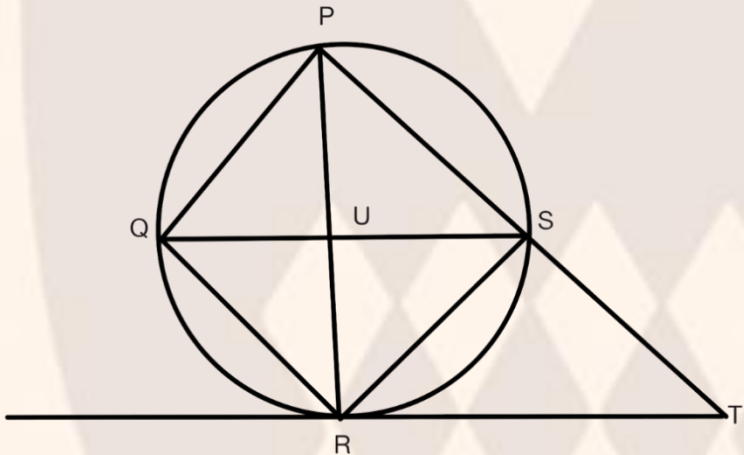
3	In the diagram, PQ is the diameter of a circle with centre O. Triangle PQR is isosceles with $PQ=PR$. The line QR meets the circle again at T. The tangent at T meets PR at S.
---	---

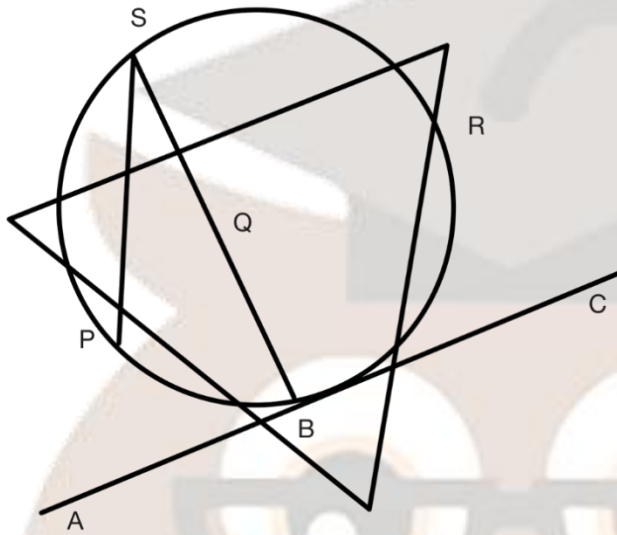
- (i) Show that ($\angle TSR = 90^\circ$.)
(ii) Explain why the circle passing through the points S,R,T has its centre at the midpoint of TR.



4 A, B and C are points on the circle with centre O and AB is the diameter. AC is produced to D and BD is a tangent to the circle at B.

- (i) Stating your reasons clearly, show that triangle ABC is similar to triangle BDC.
(ii) Given that $AC = 4.5$ cm and $BC = 6$ cm, find CD.
(iii) Determine whether $OC = \frac{1}{2} BD$.

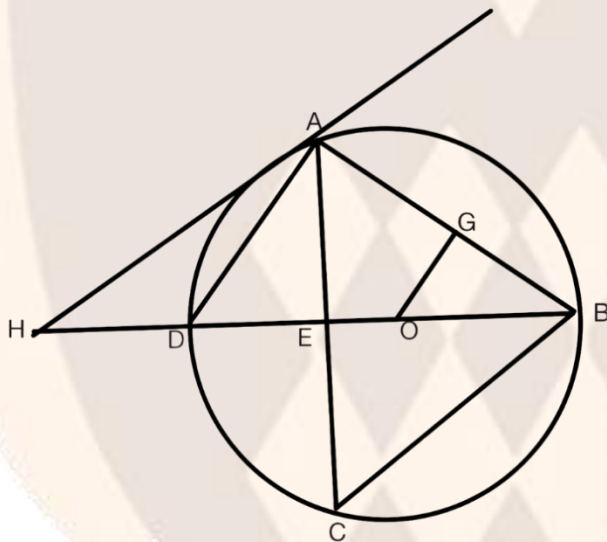
		
5	PQRS is a cyclic quadrilateral whose diagonals PR and QS meet at U. A tangent at R meets the extension of PS at T. Suppose $QR=RS$. Prove: (a) $(QS \parallel RT)$. (b) $(\triangle QUR \sim \triangle RST)$. 	
6	In the diagram, (AC') is tangent at B, SB is a chord of the circle. Given that BP and BR bisects angle SBA and SBC respectively,	



- (i) Prove that $(\angle PBR = 90^\circ)$.
- (ii) Explain why PR is a diameter of the circle.
- (iii) Show that $(\triangle SPQ \sim \triangle RBQ)$.

7

The diagram shows a circle with centre O, and four points A, B, C, D on the circumference. The line HA is tangent to the circle at A. Point G is the midpoint of chord AB. The line AD is the internal bisector of angle $(\angle CAH)$.



Prove:

- (a) OG is perpendicular to AB.
- (b) $(\angle ABD = \angle CBD)$

Answer Key:

1	(a) Similarity of $(\triangle PST)$ and $(\triangle PQR)$. 1. $(\angle SPT = \angle QPR)$. – These are the same (common) angle at P.	
---	--	--

2. Since ($AB \parallel ST$), the angle $\angle PST$ is alternate to $\angle SPA$. But AB is tangent at P, so by the alternate-segment theorem $\angle SPA = \angle PQR.$ Thus ($\angle PST = \angle PQR$.) (Alternatively one may observe ($\angle PTS = \angle TPB$) by parallel lines, and then use the alternate-segment theorem to identify ($\angle TPB = \angle PRQ$). 3. Two pairs of equal angles imply $\triangle PST \sim \triangle PQR.$ (b) Product of segments. From the similarity in (a) we have the ratio of corresponding sides: $\frac{PQ}{PS} = \frac{PR}{PT}.$ Cross-multiplying gives $PQ \times PT = PR \times PS.$ (c) Is STQR cyclic? 1. Consider ($\angle QTS$). Since T lies on the straight line through P parallel to AB, $\angle QTS = 180^\circ - \angle PTS$ (adjacent angles on a straight line). 2. From part (a) we have ($\angle PTS = \angle PRQ$). Hence $\angle QTS = 180^\circ - \angle PRQ.$ 3. Thus $\begin{aligned} &\angle QTS + \angle QRS \\ &= (180^\circ - \angle PRQ) + \angle QRS = 180^\circ \end{aligned}$ since ($\angle PRQ + \angle QRS = 180^\circ$) as they are a pair of angles in the same segment 4. In any quadrilateral WXYZ, the sum of the opposite angles satisfies

	$(\angle W + \angle Y = 360^\circ - (\angle X + \angle Z))$. Here in $STQR$, let $\angle RST + \angle RQT$ $= 360^\circ - (\angle QTS + \angle QRS)$ $= 360^\circ - 180^\circ = 180^\circ$ This is exactly the condition that a pair of opposite angles sum to (180°). Therefore, by the converse of the cyclic-quadrilateral criterion (opposite angles supplementary), $(STQR)$ is cyclic. Final Answer: (a) $(\triangle PST \sim \triangle PQR)$. (b) $(PQ \times PT = PR \times PS)$. (c) Yes. Opposite angles in $STQR$ sum to 180°, so $STQR$ is cyclic.	
2	(a) Show $(\angle BCD = 90^\circ)$. 1. Since DCP is tangent at D, the tangent–chord theorem tells us $\angle BCD$ $=$ (angle in the opposite segment) $= \angle BAD$. 2. But AD is a diameter of the circle, so $(\angle BAD)$ subtends a semicircle and hence $\angle BAD = 90^\circ$. 3. Therefore $\angle BCD = 90^\circ$. (b) Show $(DP^2 = BP^2 + DA \cdot BC)$. We shall use (i) the tangent–secant (power–of–a–point) theorem at P, and (ii) a pair of similar right triangles. 1. Tangent–secant theorem at P: The (tangent–secant) power–of–a–point says $(\text{tangent from } P)^2 = (\text{external secant}) \times (\text{whole secant}).$ Here the tangent from P to the circle is PD, and the secant is the line PAB. Thus $DP^2 = PB \times PA.$	

	Since P lies on the extension of AB, $PA = AB + BP$, so $DP^2 = PB(AB + BP)$ $= PB \times AB + PB^2.$ Hence $DP^2 - BP^2 = PB \cdot AB.$ 2. Similar right triangles showing $(PB \times AB = DA \times BC)$. • Because AD is a diameter, triangle BDA is right at D, so $(\angle BDA = 90^\circ)$. • From part (a), $(\angle BCD = 90^\circ)$. • We are also given $(\angle CBP = \angle DEB)$, and by the cyclic quadrilateral A,B,D,E, $(\angle DEB = \angle DAB)$. Hence $\angle CBP = \angle DAB$. • So in triangles $(\triangle BPC)$ and $(\triangle ADB)$ we have – both are right triangles, – they share the acute angle at B: $(\angle CBP = \angle DAB)$. Therefore $(\triangle BPC \sim \triangle ADB)$. From the similarity, $\frac{PB}{AD} = \frac{BC}{AB}$ $\Rightarrow$ $PB \cdot AB = AD \cdot BC.$ 3. Substitute (2) into (1): $DP^2 - BP^2 = PB \times AB = DA \times BC \quad \Rightarrow$ $DP^2 = BP^2 + DA \cdot BC,$ as required. Hence $DP^2 = BP^2 + DA \cdot BC.$	
3	(i) $\angle TSR = 90^\circ$. Proof sketch: • Since PQ is a diameter, the inscribed angle $\angle PTQ = 90^\circ$. • By the tangent–chord theorem, the angle between tangent TT and chord TR equals the	

	angle in the opposite segment: $\angle STR = \angle PTQ = 90^\circ$. In triangle STR, $\angle TSR + \angle STR + \angle SRT = 180^\circ$, and $\angle STR = 90^\circ$, so $\angle TSR = 90^\circ$. (ii) The circle through S, R, T has $\angle STR = 90^\circ$, so TR subtends a right angle on that circle. Hence TR must be a diameter of the circle, and its centre is the midpoint of TR.	
4	(i) Similarity of $(\triangle ABC)$ and $(\triangle BDC)$ 1. Right angle at B in $(\triangle ABC)$: Since AB is a diameter, the angle in a semicircle is a right angle. $\angle ACB = 90^\circ$. 2. Right angle at B in $(\triangle BDC)$: BD is tangent at B, and the radius OB is perpendicular to the tangent. Hence $\angle CBD = 90^\circ$. 3. Common angle at C: Both triangles share the angle ($\angle BCA = \angle DCB$). Since they have two angles equal, $\triangle ABC \sim \triangle BDC$. (ii) Finding CD From the similarity $\frac{BC}{DC} = \frac{AC}{BC}$ Rearrange to solve for DC: $ \begin{aligned} DC &= \frac{BC^2}{AC} \\ &= \frac{6^2}{4.5} \\ &= \frac{36}{4.5} \\ &= 8\text{cm.} \end{aligned} $ (iii) Checking whether $(OC = \frac{1}{2} BD)$ 1. Compute BD:	

	$(BD = BC + CD = 6 + 8 = 14)$ cm. So $\frac{1}{2} BD = 7$ cm. 2. Compute OC: Since AB is the diameter, the radius $(R = \frac{1}{2} AB)$. But in right-angled $(\triangle ABC)$, $AB = \sqrt{AC^2 + BC^2}$ $= \sqrt{4.5^2 + 6^2}$ $= \sqrt{20.25 + 36}$ $= \sqrt{56.25}$ $= 7.5$ cm. Hence $OC = R = \frac{7.5}{2} = 3.75$ cm. Since $(OC = 3.75 \neq 7)$, we conclude $OC \neq \frac{1}{2} BD$.	
5	(a) We shall show $(\angle QSR = \angle SRT)$. Then by the converse of the alternate-interior-angle test, $(QS \parallel RT)$. 1. Since $(QR = RS)$, triangle (QRS) is isosceles with vertex R. Hence its base-angles are equal: $\angle RQS = \angle QSR$. 2. By the “tangent–chord” (alternate-segment) theorem, $\angle SRT$ = the angle between tangent at R and chord RS = $\angle RQS$. 3. Therefore $\angle SRT = \angle RQS = \angle QSR$, so $\angle QSR = \angle SRT$ $\Rightarrow$ $QS \parallel RT$. (b) Now that we know $(QS \parallel RT)$, we can exhibit two pairs of equal angles to show $(\triangle QUR)$ is similar to $(\triangle RST)$. 1. Because $(QS \parallel RT)$, corresponding angles along the transversal (PR) give	

	$\angle QRP = \angle RSP$. 2. Again by the tangent–chord theorem at R, $\angle RQU = \angle TRS$. Here $\angle RQU$ is an angle in the cyclic quadrilateral and $\angle TRS$ is formed by the tangent RT and chord RS. 3. Thus in triangles QUR and RST we have $\angle QRP = \angle RSP, \angle RQU = \angle TRS$. By the Angle–Angle criterion, $(\triangle QUR \sim \triangle RST)$. – Proof completed	
6	(i) Prove that $(\angle PBR = 90^\circ)$. 1. Let $\angle ABP = x, \quad \angle CBR = y$. 2. By the alternate–segment theorem (tangent–chord angle), $\angle PRB = \angle ABP = x$ $\angle BPR = \angle CBR = y$ 3. Since BP bisects $(\angle SBA)$, $(\angle ABP = \angle PBS = x)$. Since BR bisects $(\angle SBC')$, $(\angle CBR = \angle RBS = y)$. 4. In $(\triangle PBR)$, $\angle PBR + \angle PRB + \angle RPB = 180^\circ$ $\Rightarrow$ $(x + y) + x + y = 180^\circ$ $\Rightarrow$ $2x + 2y = 180^\circ$ $\Rightarrow$ $x + y = 90^\circ$. Hence $\angle PBR = 90^\circ$.	

	(ii) Explain why PR is a diameter of the circle. • Since $(\angle PBR = 90^\circ)$, <i>triangle</i> (PBR) is a right-angle triangle with right angle at B. • A triangle inscribed in a circle is right-angled if and only if its hypotenuse is a diameter. • Therefore, PR is the diameter. (iii) Show that $(\triangle SPQ \sim \triangle RBQ)$. 1. By the alternate-segment theorem, $(\angle BRP = \angle BSP)$. 2. Vertically opposite angles give $(\angle SQP = \angle RQB)$. 3. Hence two angles equal, so by AA, $(\triangle SPQ \sim \triangle RBQ)$. (iv) It is given that $(4PQ = QR)$ and $SQ=QB$. Prove $(\left(\frac{QB}{PQ}\right)^2 = 4)$. From the similarity $(\triangle SPQ \sim \triangle RBQ)$, corresponding sides satisfy $\frac{SQ}{RQ} = \frac{PQ}{BQ} \Rightarrow SQ \times BQ = PQ \times RQ.$ Substitute $(RQ = 4PQ)$ and $(SQ = QB)$: $(QB) \times QB = PQ \times (4PQ)$ $\Rightarrow$ $QB^2 = 4PQ^2$ $\Rightarrow$ $\left(\frac{QB}{PQ}\right)^2 = 4.$	
7	(a) In $(\triangle ADB)$: - G is the midpoint of AB (given). - O is the midpoint of DB (since D and B lie on the circle and OB and OD are radii). Therefore, by the Mid-point Theorem, $GO \parallel AD$. But $(\angle DAB = 90^\circ)$ (the angle in a semicircle). Since HA is tangent, $(\angle DAB)$ is also the angle between the tangent and chord AB. Hence the line AD is perpendicular to AB. Because $(GO \parallel AD)$, it follows that $\angle GOB = 90^\circ$ $\Rightarrow$	

	$OG \perp AB$. (b) We know (AD) bisects $(\angle CAH)$. Thus $\angle DAH = \angle CAD.$ Next, by the Alternate Segment Theorem, $\angle ABD = \angle DAH.$ Also, since C and B lie on the same circle segment cut off by chord AD, $\angle CBD = \angle CAD.$ Putting these together, $\angle ABD = \angle DAH = \angle CAD = \angle CBD$ as required.	
8	1. Recall the Inscribed-Angle Theorem: An inscribed angle is half the measure of its intercepted arc. 2. Identify which arc each angle intercepts: - At A: the angle between chords AC and AD intercepts the arc CD. $\angle A = \frac{1}{2} (\widehat{CD}).$ - At B: the angle between chords BD and BE intercepts the arc DE. $\angle B = \frac{1}{2} (\widehat{DE}).$ - At C: the angle between chords CE and CA intercepts the arc EA. $\angle C = \frac{1}{2} (\widehat{EA}).$ - At D: the angle between chords DA and DB intercepts the arc AB.	

	$\angle D = \frac{1}{2} (\widehat{AB}).$ - At E: the angle between chords EB and EC intercepts the arc BC. $\angle E = \frac{1}{2} (\widehat{BC}).$ 3. Add them up: $\angle A + \angle B + \angle C + \angle D + \angle E$ $= \frac{1}{2} (\widehat{CD} + \widehat{DE} + \widehat{EA} + \widehat{AB} + \widehat{BC}).$ 4. Notice that the five arcs ($\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{DE}$, $\widehat{EA}$) together make the entire circle, i.e Total 360° Hence $\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DE} + \widehat{EA} = 360^\circ.$ Therefore $\begin{aligned} \angle A + \angle B + \angle C + \angle D + \angle E &= \frac{1}{2} (360^\circ) \\ &= 180^\circ. \end{aligned}$	
9	1. Alternate-segment theorem. In any circle, the angle between the tangent at a point and the chord through that point equals the angle in the opposite segment. - In the smaller circle at point P, let t be the tangent line. - The chord through P is PA. - By the alternate-segment theorem, $\angle(t, PA) = \angle PBA.$ Here $(\angle(t, PA))$ means the (acute) angle from line PA to the tangent t. 2. Cyclic quadrilateral XABY. The points X,A,B,Y lie on the larger circle. Therefore	

	$\angle YXA = \angle YBA$ (angles in the same segment). 3. Collinearity to identify ($\angle YBA$). Since PBY are collinear (the line PBY is straight), $\angle YBA = \angle PBA$. 4. Combine the facts. From steps 1–3: $\angle (t, PA)$ $= \angle PBA$ $= \angle YBA$ $= \angle YXA$. But ($\angle YXA$) is exactly the angle between the line XY and XA. Since PAX are collinear, XA is the same line as PA. Hence $\angle (\text{tangent at } P, PA)$ $= \angle (XY, PA)$, which means the tangent at P is parallel to XY. Conclusion. The tangent to the smaller circle at P is parallel to XY.	
--	---	--