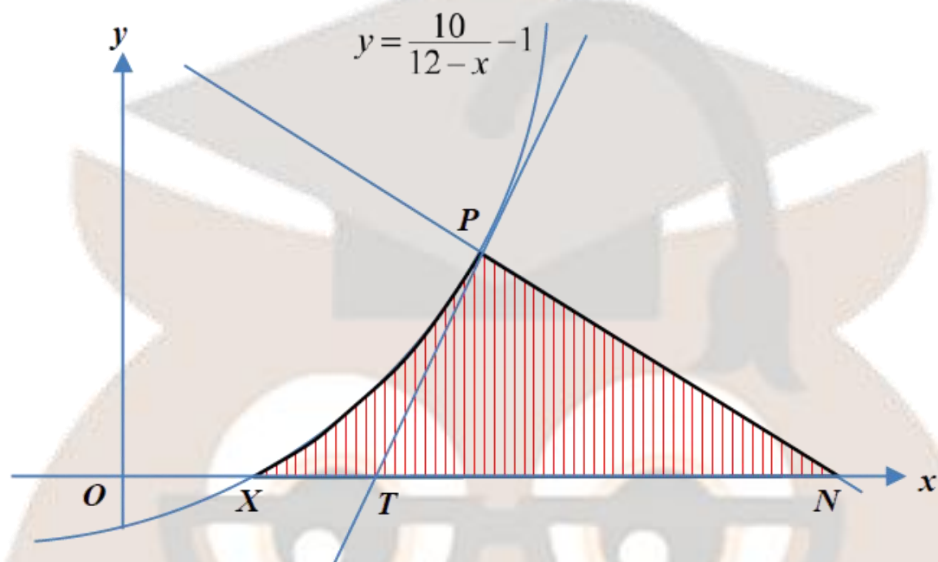


11 ESSENTIAL QUESTIONS TO MASTER

INTEGRATION

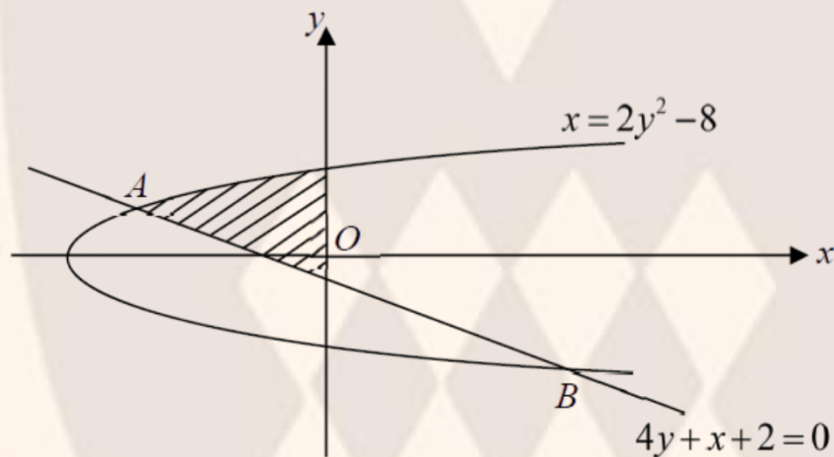
1	(a) Show that $(\sin x + \cos x)^2 = 1 + \sin 2x$. (b) Hence find, in terms of π, the value of $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx$.	
2	(a) Given that $\frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)} = \frac{A}{2x + 1} + \frac{Bx + C}{x^2 + 4},$ where A,B,C are constants, find A and B and show that C=0. (b) Differentiate $(\ln(x^2 + 4))$ with respect to (x). (c) Using the results from parts (a) and (b), find $\int \frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)} dx$.	
3	The rate of change of A with respect to r is given by $\frac{dA}{dr} = 4r + 7$. If A = 12 when r = 1, find A in terms of r.	
4	(a) Given that $\frac{dy}{dx} = 1 - \sin 2x$ find an expression for y if $(y = \frac{3}{2})$ when $(x = 0)$. (b) Given that $\frac{dy}{dx} = 1 + \frac{1}{2} \cos 2x$ find an expression for (y) if $(y = 3)$ when $(x = \frac{\pi}{4})$.	
5	(i) Given that $y = 3x^2 \ln(4x)$, find $\frac{dy}{dx}$ (ii) Hence find $\int x \ln(4x) dx$.	
6	The equation of a curve is $y = A e^{2x} + B e^{-x}$, where A and B are constants. The point P(0,4) lies on the curve and the gradient of the tangent to the curve at P is -1. (a) Find the values of A and B.	

	(b) Using your values of A and B from part (a), obtain an expression for $\int y \, dx$ and hence evaluate $\int_0^1 y \, dx$ correct to one decimal place.	
7	It is given that $\int f(x) \, dx = \sin x + k \cos 2x + C$ where C is a constant of integration, and that $\int_0^{\frac{\pi}{6}} f(x) \, dx = \frac{3}{4}.$ (i) Show that $(k = -\frac{1}{2})$. (ii) Find $(f(x))$. (iii) Find the equation of the normal to the curve $(y = f(x))$ at the point where $(x = \frac{\pi}{6})$	
8	(i) Show that $\frac{d}{dx} \left(\frac{\ln x}{x^2} \right) = \frac{1}{x^3} - \frac{2 \ln x}{x^3}$. (ii) Find $(\int \frac{\ln x}{x^3} \, dx)$. (iii) Given that the curve $(y = f(x))$ passes through $(1, \frac{3}{4})$ and satisfies $f'(x) = \frac{\ln x}{x^3}$ find $(f(x))$.	
9	A particle P travels in a straight line so that its displacement s m from O at time t s is modelled by $s = \frac{1}{3}t^3 - 5t^2 - 3.$ (a) Find the value of t when particle P return to its initial position. (b) Find the minimum velocity of particle P. (c) Time when another particle (Q) is instantaneously at rest Particle (Q) has velocity $v = 24 \left(e^{-\frac{t}{6}} - e^{-1} \right)$ Find the value of t at which the particle Q is instantaneously at rest. (d) Find the total distance travelled by particle Q for the first 9 seconds.	
10	The diagram shows part of the curve $y = \left(\frac{10}{12-x} \right) - 1$ passing through the point $(P(2k, k - 1))$, where k is a constant. The curve meets the x-axis at the point X. The tangent and normal at P meet the x-axis at the points T and N respectively.	



- (a) Find the equation of the normal to the curve at P .
(b) Find the exact area of the shaded region bounded by the curve from X to P , the line segment PN , and the x -axis from N back to X .

- 11 In the diagram, the parabola $x = 2y^2 - 8$ meets the straight line $4y + x + 2 = 0$ at two points A and B .



- (i) Find the coordinates of A and B .
(ii) Find the area of the shaded region.

Answer Key:

1	(a) Algebraic identity We expand the left-hand side: $(\sin x + \cos x)^2$ $= \sin^2 x + 2\sin x \cos x + \cos^2 x.$ Recall the Pythagorean identity ($\sin^2 x + \cos^2 x = 1$), and the double-angle identity ($\sin 2x = 2\sin x \cos x$) Hence, $\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + \sin 2x$ is the answer. (b) Definite integral We now compute $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx$ $= \int_0^{\frac{\pi}{2}} (1 + \sin 2x) dx \text{ by using the result of part (a). Split the integral:}$ $\int_0^{\frac{\pi}{2}} 1 dx + \int_0^{\frac{\pi}{2}} \sin 2x dx$ 1. $\int_0^{\frac{\pi}{2}} 1 dx$ $= [x]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{2}$ 2. $\int_0^{\frac{\pi}{2}} \sin 2x dx \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{2}} - \frac{1}{2} (\cos \pi - \cos 0) - \frac{1}{2} (-1 - 1)$ $= -\frac{1}{2} (-2)$ $= 1.$ Adding these, $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx = \frac{\pi}{2} + 1.$	
2	(a) Partial-fraction decomposition We seek constants A,B,C such that $\frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)}$	

$= \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}.$ Clear denominators by multiplying both sides by $((2x+1)(x^2+4))$: $3x^2 + 4x - 20 = A(x^2 + 4) + (Bx + C)(2x + 1).$ Expand the right hand side: $Ax^2 + 4A + (Bx + C)(2x + 1) = Ax^2 + 4A + 2Bx^2 + Bx + 2Cx + C$ $= (A + 2B)x^2 + (B + 2C)x + (4A + C).$ Compare coefficients with $(3x^2 + 4x - 20)$: - (x^2)-term: $(A + 2B = 3).$ - (x)-term: $(B + 2C = 4).$ - constant term: $(4A + C = -20).$ We are told to show $(C = 0)$. Suppose $(C = 0)$. Then the second equation $(B + 2C = 4)$ gives $B + 2 \times 0 = 4$ $\Rightarrow B = 4.$ The first equation $(A + 2B = 3)$ becomes $A + 2 \times 4 = 3$ $\Rightarrow A + 8 = 3$ $\Rightarrow A = -5.$ Finally check the constant term: $4A + C = 4(-5) + 0 = -20,$ which matches. Hence $A = -5, \quad B = 4, \quad C = 0.$ (b) Differentiation $\frac{d}{dx} [\ln(x^2 + 4)]$ $= \frac{1}{x^2 + 4} \times \frac{d}{dx} (x^2 + 4)$ $= \frac{2x}{x^2 + 4}.$ (c) Indefinite integral From part (a) we have	
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$$\begin{aligned} & \frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)} \\ &= \frac{-5}{2x + 1} + \frac{4x + 0}{x^2 + 4} \\ &= -5 \times \frac{1}{2x + 1} + 4 \times \frac{x}{x^2 + 4}. \end{aligned}$$

Hence

$$\begin{aligned} & \int \frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)} dx \\ &= -5 \int \frac{dx}{2x + 1} \\ &+ 4 \int \frac{x}{x^2 + 4} dx. \end{aligned}$$

Evaluate each:

$$\begin{aligned} 1. & \int \frac{dx}{2x+1} = \frac{1}{2} \ln|2x + 1| + C_1. \\ 2. & \int \frac{x}{x^2+4} dx. \text{ Let } (u = x^2 + 4), \text{ then } (du = 2x dx), \text{ so} \\ & x dx = \frac{1}{2} du \end{aligned}$$

Hence

$$\begin{aligned} & \int \frac{x}{x^2 + 4} dx \\ &= \int \frac{1}{u} \times \frac{1}{2} du \\ &= \frac{1}{2} \ln|u| + C_2 \\ &= \frac{1}{2} \ln(x^2 + 4) + C_2. \end{aligned}$$

Putting these together,

$$\begin{aligned} & \int \frac{3x^2 + 4x - 20}{(2x + 1)(x^2 + 4)} dx \\ &= -5 \times \frac{1}{2} \ln|2x + 1| \\ &+ 4 \times \frac{1}{2} \ln(x^2 + 4) \\ &+ C \\ &= -\frac{5}{2} \ln|2x + 1| \\ &+ 2 \ln(x^2 + 4) \\ &+ C, \end{aligned}$$

where C is the arbitrary constant of integration.
That completes all parts.

3	We are given $\frac{dA}{dr} = 4r + 7$. To find $(A(r))$, integrate both sides with respect to (r): $A(r) = \int (4r + 7) dr$. Compute the integral: $\int (4r + 7) dr$ $= \int 4r dr + \int 7 dr$ $= 4 \times \frac{r^2}{2} + 7r + C$ $= 2r^2 + 7r + C$, where C is the constant of integration. Determine C from the initial condition We know $(A = 12)$ when $(r = 1)$. Substitute into $(A(r) = 2r^2 + 7r + C)$: $12 = 2(1)^2 + 7(1) + C$ $\Rightarrow$ $12 = 2 + 7 + C$ $\Rightarrow$ $12 = 9 + C$ $\Rightarrow$ $C = 3.$ Final answer $A(r) = 2r^2 + 7r + 3.$	
4	(a) Integrate $\frac{dy}{dx} = 1 - \sin 2x$ 1. Write the integral: $y = \int (1 - \sin 2x) dx$ 2. Integrate term by term. $\int 1 dx = x,$ $\int (-\sin 2x) dx = -\left(-\frac{1}{2} \cos 2x\right) = \frac{1}{2} \cos 2x \text{ so}$ $y = x + \frac{1}{2} \cos 2x + C$	

	3. Determine (C) from the initial condition ($y(0) = \frac{3}{2}$): $y(0) = 0 + \frac{1}{2} \cos 0 + C = \frac{1}{2} \times 1 + C = \frac{1}{2} + C = \frac{3}{2} \Rightarrow C = 1.$ 4. Final answer: $y = x + \frac{1}{2} \cos 2x + 1.$ (b) Integrate $\frac{dy}{dx} = 1 + \frac{1}{2} \cos 2x$ 1. Set up the integral: $y = \int \left(1 + \frac{1}{2} \cos 2x \right) dx$ 2. Integrate: $\int 1 dx = x,$ $\int \frac{1}{2} \cos 2x dx = \frac{1}{2} \times \frac{1}{2} \sin 2x = \frac{1}{4} \sin 2x \text{ Hence}$ $y = x + \frac{1}{4} \sin 2x + C$ 3. Use ($y\left(\frac{\pi}{4}\right) = 3$). Note ($\sin\left(2 \times \frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1$): $3 = \frac{\pi}{4} + \frac{1}{4} \times 1 + C$ $= \frac{\pi}{4} + \frac{1}{4} + C$ $\Rightarrow C = 3 - \left(\frac{\pi}{4} + \frac{1}{4}\right)$ $= \frac{12 - (\pi + 1)}{4} = \frac{11 - \pi}{4}.$ 4. Final answer: $y = x + \frac{1}{4} \sin 2x + \frac{11 - \pi}{4}.$	
5	(i) Differentiation We have $y = 3x^2 \ln(4x)$. Differentiate by the product rule: if ($u = 3x^2$) and ($v = \ln(4x)$), then $u' = 6x,$ $v' = \frac{d}{dx} [\ln(4x)]$	

$$= \frac{1}{4x} \times 4 = \frac{1}{x}.$$

So

$$\frac{dy}{dx}$$

$$= u'v + uv'$$

$$= 6x \ln(4x) + 3x^2 \times \frac{1}{x}$$

$$= 6x \ln(4x) + 3x.$$

Answer for (i):

$$\frac{dy}{dx} = 6x \ln(4x) + 3x.$$

(ii) Integration

We now wish to find

$$\int x \ln(4x) dx.$$

Notice from part (i) that

$$\frac{dy}{dx} = 6x \ln(4x) + 3x$$

$\Rightarrow$

$$6x \ln(4x) = \frac{dy}{dx} - 3x.$$

Hence

$$x \ln(4x)$$

$$= \frac{1}{6} \frac{dy}{dx} - \frac{1}{2} x.$$

Integrate both sides with respect to x:

$$\int x \ln(4x) dx$$

$$= \int \left(\frac{1}{6} y' - \frac{1}{2} x \right) dx$$

$$= \frac{1}{6} y - \frac{1}{2} \times \frac{x^2}{2} + C$$

$$= \frac{1}{6} (3x^2 \ln(4x)) - \frac{x^2}{4} + C$$

Simplify to form answer:

$$\int x \ln(4x) dx = \frac{1}{2} x^2 \ln(4x) - \frac{1}{4} x^2 + C$$

6	(a) Finding A and B 1. Since P(0,4) lies on the curve, when $x=0$ we have $y(0) = A e^0 + B e^0 = A + B = 4.$ So $A + B = 4$. 2. Compute the derivative: $y = A e^{2x} + B e^{-x}$ $\Rightarrow y' = 2A e^{2x} - B e^{-x}.$ At $(x = 0)$, the gradient $(y'(0))$ is given to be (-1), so $y'(0) = 2A \cdot e^0 - B \cdot e^0 = 2A - B = -1.$ 3. Solve the system $\begin{aligned} A + B &= 4 \\ 2A - B &= -1. \end{aligned}$ Adding the two equations gives $3A = 3$, so $A = 1.$ Then from $(A + B = 4)$ we get $B = 4 - A = 3.$ Answer (a): $(A = 1, B = 3)$. (b) Finding $(\int y \, dx)$ and the definite integral 1. Substitute $(A=1), (B=3)$ into (y): $y = e^{2x} + 3 e^{-x}.$ 2. Integrate term by term:	
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	$\int y \, dx$ $= \int e^{2x} \, dx + 3 \int e^{-x} \, dx$ $= \frac{1}{2} e^{2x} - 3 e^{-x} + C.$ Hence $\boxed{\int y \, dx = \frac{1}{2} e^{2x} - 3 e^{-x} + C}.$ 3. Now evaluate the definite integral from 0 to 1: $\int_0^1 y \, dx$ $= \left[\frac{1}{2} e^{2x} - 3 e^{-x} \right]_{x=0}^{x=1}$ $= \left(\frac{1}{2} e^2 - \frac{3}{e} \right) - \left(\frac{1}{2} \times 1 - 3 \times 1 \right).$ Simplify the second bracket: $(\frac{1}{2} - 3 = -\frac{5}{2})$. So $\int_0^1 y \, dx$ $= \frac{1}{2} e^2 - \frac{3}{e} + \frac{5}{2}.$ Numerically, $\frac{1}{2} e^2 \approx 3.6945, \quad \frac{3}{e} \approx 1.1030, \quad \frac{5}{2} = 2.5,$ So $\int_0^1 y \, dx \approx 3.6945 - 1.1030 + 2.5 = 5.0915 \approx 5.1.$ Answer (b): $(\int y \, dx = \frac{1}{2} e^{2x} - 3 e^{-x} + C)$ and $(\int_0^1 y \, dx \approx 5.1)$ (to one decimal place).	
7	(i) Compute the definite integral using the given antiderivative: $\int_0^{\pi/6} f(x) \, dx$ $= [\sin x + k \cos 2x]_0^{\pi/6} \left(\sin \frac{\pi}{6} + k \cos \frac{\pi}{3} \right) - (\sin 0 + k \cos 0)$ Now $(\sin(\frac{\pi}{6}) = \frac{1}{2}), (\cos(\frac{\pi}{3}) = \frac{1}{2}), (\sin 0 = 0), (\cos 0 = 1)$. So	

$$\int_0^{\frac{\pi}{6}} f(x) dx = \left(\frac{1}{2} + k \times \frac{1}{2}\right) - (0 + k \times 1) = \frac{1}{2} + \frac{k}{2} - k = \frac{1}{2} - \frac{k}{2}$$

But we are told this equals to $\frac{3}{4}$. Hence

$$\frac{1}{2} - \frac{k}{2} = \frac{3}{4}$$

$\Rightarrow$

$$-\frac{k}{2} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

$\Rightarrow$

$$k = -\frac{1}{2}$$

(ii) Differentiate the antiderivative to recover $(f(x))$:

$$\begin{aligned} \int f(x) dx &= \sin x + k \cos 2x + C \rightarrow f(x) = \frac{d}{dx}(\sin x + k \cos 2x) \\ &= (\cos x - 2k \sin 2x) \end{aligned}$$

Substitute $(k = -\frac{1}{2})$:

$$f(x) = \cos x - 2\left(-\frac{1}{2}\right) \sin 2x \cos x + \sin 2x.$$

Since $(\sin 2x = 2 \sin x \cos x)$ we may also write $f(x) = \cos x + 2 \sin x \cos x$.

(iii) We want the normal line to $(y = f(x))$ at $(x = \frac{\pi}{6})$.

1. Compute the slope of the tangent, $(m_{\tan} = f'(x))$.

$$\begin{aligned} f(x) &= \cos x + \sin 2x \\ f'(x) &= -\sin x + 2 \cos 2x. \end{aligned}$$

At $(x = \frac{\pi}{6})$:

$$\begin{aligned} f'\left(\frac{\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right) + 2 \cos\left(\frac{\pi}{3}\right) \\ &= -\frac{1}{2} + 2 \times \frac{1}{2} = -\frac{1}{2} + 1 = \frac{1}{2} \end{aligned}$$

So the tangent slope is $(+\frac{1}{2})$. Thus the normal slope is

	$m_{\text{normal}} = -\frac{1}{m_{\text{tan}}}$ $= -\frac{1}{\frac{1}{2}}$ $= -2.$ 2. Find the point (x, y) on the curve at $(x = \frac{\pi}{6})$: $y = f\left(\frac{\pi}{6}\right)$ $= \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)$ $= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$ $= \sqrt{3}.$ So the point is $\left(\frac{\pi}{6}, \sqrt{3}\right)$. 3. Write the normal-line equation in point-slope form: $y - \sqrt{3} = -2\left(x - \frac{\pi}{6}\right).$ This is the required normal. Final Answers: (i) $(k = -\frac{1}{2}.)$ (ii) $(f(x) = \cos x + \sin 2x.)$ (iii) The equation of the normal at $x = \frac{\pi}{6}$ is $y - \sqrt{3} = -2\left(x - \frac{\pi}{6}\right)$	
8	(i) Differentiation We write $\frac{\ln x}{x^2} = (\ln x) x^{-2}.$	

Differentiate by the product rule:

$$\begin{aligned} & \frac{d}{dx} [(\ln x) x^{-2}] \\ &= (\ln x)' x^{-2} + (\ln x) (x^{-2})' \\ &= \frac{1}{x} x^{-2} + \ln x (-2)x^{-3}. \end{aligned}$$

Since $(\frac{1}{x} x^{-2} = x^{-3})$, this becomes

$$\begin{aligned} & x^{-3} - 2 \ln x x^{-3} \\ &= \frac{1}{x^3} - \frac{2 \ln x}{x^3}, \end{aligned}$$

as required.

(ii) Integration

We wish to compute

$$\begin{aligned} & \int \frac{\ln x}{x^3} dx \\ &= \int (\ln x) x^{-3} dx. \end{aligned}$$

Use integration by parts with

$$u = \ln x, \quad dv = x^{-3} dx$$

$\Rightarrow$

$$du = \frac{1}{x} dx, \quad v = \int x^{-3} dx = -\frac{1}{2x^2}.$$

Hence

$$\begin{aligned} & \int (\ln x) x^{-3} dx \\ &= u v - \int v du \\ &= -\frac{\ln x}{2x^2} \\ & \quad - \int \left(-\frac{1}{2x^2}\right) \frac{1}{x} dx \\ &= -\frac{\ln x}{2x^2} \\ & \quad + \frac{1}{2} \int x^{-3} dx. \end{aligned}$$

But $(\int x^{-3} dx = -\frac{1}{2x^2})$, so

	$\int \frac{\ln x}{x^3} dx$ $= -\frac{\ln x}{2x^2}$ $+ \frac{1}{2} \left(-\frac{1}{2x^2} \right)$ $= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$ We can combine: $\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$ (iii) Finding $f(x)$ Since $(f'(x) = \frac{\ln x}{x^3})$, we have by part (ii) $f(x) = \int f'(x) dx$ $= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$ Use the condition $(f(1) = \frac{3}{4})$. When $(x = 1)$, $(\ln 1 = 0)$, so $f(1) = -\frac{0}{2 \times 1^2} - \frac{1}{4 \times 1^2} + C$ $= -\frac{1}{4} + C$ $= \frac{3}{4} \Rightarrow C = 1$ Therefore $f(x) = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + 1$	
9	(a) Find the value of t when particle P return to its initial position. - Initial position is at $(t = 0)$, namely $(s = -3)$. - To find when it returns to $(s = -3)$, set $\frac{1}{3}t^3 - 5t^2 - 3 = -3.$ - Simplify: $\frac{1}{3}t^3 - 5t^2 = 0$ $\Rightarrow$ $t^2 \left(\frac{1}{3}t - 5 \right) = 0.$ - Hence $(t = 0)$ or $(\frac{1}{3}t - 5 = 0 \Rightarrow t = 15)$.	

- Answer is ($t = 15$) s.

(b) Find the minimum velocity of particle P .

1. First find velocity ($v = \frac{ds}{dt}$).

$$\begin{aligned}
 s &= \frac{1}{3}t^3 - 5t^2 - 3 \\
 \Rightarrow \\
 v &= t^2 - 10t.
 \end{aligned}$$

2. To find the minimum, complete the square or use calculus.

- Method 1 (complete the square):

$$\begin{aligned}
 v &= t^2 - 10t \\
 &= (t - 5)^2 - 25,
 \end{aligned}$$

so minimum of (v) is $(-25) \text{ m/s}$, attained at ($t = 5$).

- Method 2 (calculus):

$$\begin{aligned}
 \frac{dv}{dt} &= 2t - 10 = 0 \\
 \Rightarrow t &= 5,
 \end{aligned}$$

$$v(5) = 5^2 - 10 \cdot 5 = 25 - 50 = -25 \text{ m/s},$$

and ($\frac{d^2v}{dt^2} = 2 > 0$), confirming a minimum.

- Minimum velocity = $(-25) \text{ m/s}$.

(c) Time when another particle (Q) is instantaneously at rest

Particle (Q) has velocity

$$v = 24 \left(e^{-\frac{t}{6}} - e^{-1} \right).$$

Find the value of t at which the particle Q is instantaneously at rest.

Set ($v = 0$):

$$e^{-\frac{t}{6}} - e^{-1} = 0$$

$\Rightarrow$

$$e^{-\frac{t}{6}} = e^{-1}$$

$\Rightarrow$

$$-\frac{t}{6} = -1$$

$\Rightarrow$ $t = 6 \text{ s.}$ - Answer: ($t = 6$) s. (d) Find the total distance travelled by particle Q for the first 9 seconds. 1. Find displacement function $v = \frac{ds}{dt}$ $= 24 \left(e^{-\frac{t}{6}} - e^{-1} \right).$ Integrate: $s(t)$ $= \int 24 \left(e^{-\frac{t}{6}} - e^{-1} \right) dt$ $= 24 \left(-6 e^{-\frac{t}{6}} - e^{-1} t \right) + C$ $= -144 e^{-\frac{t}{6}} - 24 e^{-1} t + C.$ Use ($s(0) = 0$) to find (C): $0 = -144 e^0 - 24 e^{-1} \times 0 + C$ $\Rightarrow$ $C = 144.$ Hence $s(t) = 144 - 144 e^{-\frac{t}{6}} - 24 e^{-1} t.$ 2. Distance from ($t = 0$) to ($t = 6$) $s(6) = 144 - 144 e^{-1} - 24 e^{-1} \cdot 6$ $= 144 - 144 e^{-1} - 144 e^{-1}$ $= 144 - 288 e^{-1}$ $\approx 38.051 \text{ m.}$ In this interval ($v > 0$), so distance = displacement = (38.051) m. 3. Distance from ($t = 6$) to ($t = 9$)	
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	$s(9) = 144 - 144e^{-\frac{9}{6}} - 24e^{-1} \cdot 9$ $= 144 - 144e^{-\frac{3}{2}} - 216e^{-1}$ $\approx 32.407 \text{ m.}$ Here (v) is negative (particle reverses), so the displacement from 6 to 9 s is $s(9) - s(6) \approx 32.407 - 38.051 = -5.644 \text{ m,}$ but the distance travelled is $(+5.644) \text{ m.}$ 4. Total distance in first 9 s $38.051 + 5.644 \approx 43.695 \text{ m,}$ or to 3 s.f. $(43.7) \text{ m.}$	
10	(a) 1. Write the curve and label it equation (1): $y = 10 \div (12 - x) - 1 \quad \dots[1]$ 2. Substitute the point P, which has coordinates $(2k, 4)$, into [1]: $4 = 10 \div (12 - 2k) - 1$ $\Rightarrow 4 + 1 = 10 \div (12 - 2k)$ $\Rightarrow 5 = 10 \div (12 - 2k)$ $\Rightarrow 12 - 2k = 10 \div 5 = 2$ $\Rightarrow 2k = 12 - 2 = 10$ $\Rightarrow k = 5.$ Hence $P = (2k, 4) = (10, 4)$. 3. Differentiate [1] to get $\frac{dy}{dx}$: $y = 10(12 - x)^{-1} - 1$ $\Rightarrow \frac{dy}{dx} = 10(-1)(12 - x)^{-2}(-1)$ $\Rightarrow \frac{dy}{dx} = 10 \div (12 - x)^2 \quad \dots[2]$ 4. Find the slope of the tangent at $P(10, 4)$ by substituting $x = 10$ into [2]: $\frac{dy}{dx} \Big _{(10)} = 10 \div (12 - 10)^2 = 10 \div 2^2 = 10 \div 4 = 5 \div 2.$ So the tangent slope $m_t = \frac{5}{2}$, and the normal slope is $m_n = -\frac{1}{\frac{5}{2}} = -\frac{2}{5}$.	

5. Equation of the normal through P(10, 4) with slope $-\frac{2}{5}$:

$$\begin{aligned} y - 4 &= \left(-\frac{2}{5}\right)(x - 10) \\ \Rightarrow y - 4 &= \left(-\frac{2}{5}\right)x + 4 \\ \Rightarrow y &= \left(-\frac{2}{5}\right)x + 8 \quad \dots[3] \end{aligned}$$

(b) Find x-coordinate of X. Since X lies on the curve and on the x-axis, its coordinates are $((x_X, 0))$. Substitute into the curve equation

$$y = \frac{10}{12 - x} - 1,$$

i.e. set $(y = 0)$ and $(x = x_X)$:

$$\begin{aligned} 0 &= \frac{10}{12 - x_X} - 1 \\ \Rightarrow \frac{10}{12 - x_X} &= 1 \\ \Rightarrow 10 &= 12 - x_X \\ \Rightarrow x_X &= 2. \end{aligned}$$

2. Find x-coordinate of N. The normal at P was found to be

$$y = -\frac{2}{5}x + 8$$

[equation (3)].

Since N lies on this normal and on the x-axis, $(x_N, 0)$ satisfies

$$\begin{aligned} 0 &= -\frac{2}{5}x_N + 8 \\ \Rightarrow \frac{2}{5}x_N &= 8 \\ \Rightarrow x_N &= 20. \end{aligned}$$

3. Compute the exact area of the shaded region. One way is to split it into

(i) the area under the curve $(y = \frac{10}{12-x} - 1)$ from $(x = 2)$ to $(x = 10)$, plus

	(ii) the triangular area between the tangent-intercept T at ($x = 10$), the normal-intercept N at ($x = 20$), and the x-axis. (i) $(\int_2^{10} (\frac{10}{12-x} - 1) dx)$. An antiderivative is $\int (\frac{10}{12-x} - 1) dx$ $= -10 \ln 12-x - x,$ so $\int_2^{10} (\frac{10}{12-x} - 1) dx$ $= [-10 \ln(12-x) - x]_{x=2}^{x=10}$ $= [-10 \ln(2) - 10] - [-10 \ln(10) - 2]$ $= 10 \ln \frac{12-2}{12-10} + (2-10)$ $= 10 \ln 5 - 8.$ (ii) The triangle has base from ($x = 10$) to ($x = 20$), so base length ($20 - 10 = 10$), and height equal to the y-value of the tangent at $x=10$. But the tangent meets the x-axis at $T(10,0)$, so the height is ($y_P = 4$). Thus its area is $\frac{1}{2} (20 - 10) \times 4 = \frac{1}{2} \times 10 \times 4 = 20.$ Adding (i) and (ii) gives $(10 \ln 5 - 8) + 20$ $= 10 \ln 5 + 12.$ Hence the exact shaded area is <u>$10 \ln 5 + 12$</u>. Alternative combined-integral approach (same result): Area $= \int_2^{10} (\frac{10}{12-x} - 1) dx + \int_{10}^{20} (-\frac{2}{5}x + 8) dx.$ Computing each definite integral and summing again yields $(10 \ln 5 + 12)$.	
11	(i) Find the coordinates of A and B.	

We substitute $(x = 2y^2 - 8)$ into the line:

$$4y + (2y^2 - 8) + 2 = 0$$

$\Rightarrow$

$$2y^2 + 4y - 6 = 0$$

$\Rightarrow$

$$y^2 + 2y - 3 = 0$$

$\Rightarrow$

$$(y + 3)(y - 1) = 0.$$

Hence

$$y = -3 \quad \text{or} \quad y = 1.$$

When $(y = -3)$, the x-coordinate is

$$x = 2(-3)^2 - 8 = 18 - 8 = 10,$$

so one intersection is $B = (10, -3)$.

When $y = 1$, the x-coordinate is

$$x = 2(1)^2 - 8 = 2 - 8 = -6,$$

so the other intersection is $A = (-6, 1)$.

(ii) Find the area of the shaded region.

1. First note where each curve meets the y-axis $x=0$.

– On the parabola $(x = 2y^2 - 8)$, setting $(x = 0)$ gives

$$2y^2 - 8 = 0$$

$\Rightarrow$

$$y^2 = 4$$

$\Rightarrow$

$$y = \pm 2.$$

– On the line $(4y + x + 2 = 0)$, setting $(x = 0)$ gives

$$4y + 2 = 0$$

$\Rightarrow$

$$y = -\frac{1}{2}$$

2. Area of the shaded region

$\int (2y^2 - 8) dy$ $= \frac{2}{3}y^3 - 8y,$ $\int_1^2 (2y^2 - 8) dy + \left(\frac{1}{2} \times \frac{3}{2} \times 6\right)$ $= \left[\frac{2}{3}y^3 - 8y\right]_1^2 + 4\frac{1}{2}$ $= \left[\left(\frac{16}{3} - 16\right) - \left(\frac{2}{3} - 8\right) + 4\frac{1}{2}\right]$ $= 1\frac{1}{6} \text{ units}^2$	
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