

21 ESSENTIAL QUESTIONS TO MASTER

DIFFERENTIATION

1	Differentiate the function $y = (3x - 1)\sqrt{x + 5}$. Find $\frac{dy}{dx}$	
2	Differentiate $y = \frac{4x+3}{\sqrt{x+5}}$ Find $\frac{dy}{dx}$	
3	Differentiate $y = 4 \sin^2(2x)$, $x = \frac{\pi}{6}$ $y = 4 \sin^2(2x)$	
4	Given that $y = 3 \sin(3x) + \cos(3x)$ prove that $\frac{d^2y}{dx^2} = -9y$.	
5	Differentiate with respect to x . $\sin(3x)(\tan^2 x + 1)$	
6	Find the x-coordinates of the points on the curve $y = \sqrt{\frac{1-x}{x^2+3}}$ for which $\frac{dy}{dx} = 0$	
7	Show that $\frac{d}{dx} \left(\frac{\ln x}{3x} \right) = \frac{1 - \ln x}{3x^2}$	
8	Given $h(x) = \ln(3x + 4)^5$. Show that $3h'(x) + (3x + 4)h''(x) = 0$.	
9	The equation of a curve is $y = \frac{a}{x} + bx - 1$, where a and b are constants. The normal to the curve at the point $(Q(1, -1))$ is parallel to the line $4y - x = 20$.	

	This normal meets the curve again at point P. (i) Find the values of a and b. (ii) Find the coordinates of point P.	
10	(a) Differentiate $\left(\tan^3\left(\frac{\pi}{3} + 3x\right)\right)$ with respect to (x) . (b) A curve is defined by $y = 4\cos(3x) + \sin x$. Find the gradient (y') of the curve when $(x = \frac{\pi}{6})$. Give your answer in exact form.	
11	A curve has equation $y = (ax - 2)\ln x$, where $(x > 0, x \neq \frac{2}{a})$, and (a) is a positive constant. The normal to the curve at the point where it crosses the x -axis is parallel to the line $x + 4y - 3 = 0$. Find the value of a .	
12	Find the range of values of x for which the gradient of the graph $y = x^4 - 2x^3 - x^2 + x + 5$ is increasing.	
13	A curve has equation $y = \frac{5x-2}{3x+1}$, where $(x \neq -\frac{1}{3})$. Show that y is always increasing as x increases.	
14	Given that $y = -\frac{1}{4}x^3 + x^2 - 5x + 1$. Show that for every real x , y is a strictly decreasing function of x .	
15	The equation of a curve is $y = \frac{4x^2}{3-2x}$. (i) Find an expression for $\frac{dy}{dx}$ (ii) If x is changing at a constant rate of 0.06 units per second, find the rate of change of y when $x = 4$.	
16	The equation of a curve is given by $y = \ln\left(\frac{7x}{5x+2}\right)$.	

	(i) Find $\frac{dy}{dx}$, expressing your answer as a single fraction. (ii) Suppose y is increasing at 0.4 units per second. Find the rate at which x is changing at the point where the curve crosses the x -axis.							
17	It is given that (x) and (y) are related by the equation $y = \frac{8}{x^2} + \frac{3}{x}.$ Find the rate of change of (x) when $(y = 3.5)$, given that the rate of change of (y) is $(1.6)units\ s^{-1}$.							
18	Given the expression $f(x) = 2x^3 - 3x^2 - 12x + 6$, find the coordinates of any turning points and determine their nature.							
19	Find the coordinates of the turning point of the curve $y = \frac{16x^3 + 1}{2x^2}$ and determine whether y is a maximum or a minimum at this point.							
20	A tennis club makes three equally sized tennis courts, positioned next to each other as shown in the diagram below. Each tennis court is rectangular and has sides of length (x) m and (y) m. The lines in the diagram represent wire netting. The total length of wire netting used is 288 m. <table border="1" data-bbox="259 1102 1339 1213" style="margin: 10px auto; width: 60%;"> <tr> <td style="height: 40px; vertical-align: top;">ym</td><td></td><td></td></tr> <tr> <td></td><td style="text-align: center;">xm</td><td></td></tr> </table> (i) Show that the total area, Am^2, of the three tennis courts is given by $A = 216x - \frac{9}{2}x^2.$ (ii) Given that (x) can vary, find the dimensions (x, y) of each tennis court that make A a maximum. (You are not required to show that A is a maximum.)	ym				xm		
ym								
	xm							
21	A curve has the equation $y = x^3 - 3x^2 - 24x + c$, where c is a constant. (i) Find the x-coordinates of the two stationary points on the curve. (ii) Hence find the value of c for which the minimum point of the curve lies on the $(x) - axis$.							

Answer Key:

1	$\frac{dy}{dx}$ $= \frac{9x - 3 + (3x - 1)}{2\sqrt{x+5}}$ $= \frac{9x - 3 + 3x - 1}{2\sqrt{x+5}}$ $= \frac{12x - 4}{2\sqrt{x+5}}$ $= \frac{2\sqrt{x+5}}{6x - 2}$ $= \frac{1}{\sqrt{x+5}}$
2	$y = (4x + 3)(x + 5)^{-\frac{1}{2}}$ Then by the product (or quotient) rule, $\frac{dy}{dx} = 4(x + 5)^{-\frac{1}{2}} + (4x + 3)\left(-\frac{1}{2}\right)(x + 5)^{-\frac{3}{2}}$ Put everything over a common denominator $(2(x + 5)^{\frac{3}{2}})$: $\frac{dy}{dx} = \frac{8(x+5) - (4x+3)}{2(x+5)^{\frac{3}{2}}}$ $= \frac{8x+40 - 4x-3}{2(x+5)^{\frac{3}{2}}}$ $= \frac{4x+37}{2(x+5)^{\frac{3}{2}}}$
3	Differentiate using the chain rule: $\frac{dy}{dx} = 4[2 \sin(2x) \cos(2x) (2)]$ $= 16 \sin(2x) \cos(2x)$ When $(x = \frac{\pi}{6})$: $\frac{dy}{dx} = 16 \sin\left(2 \times \frac{\pi}{6}\right) \cos\left(2 \times \frac{\pi}{6}\right)$ $= 16 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$ $= 16 \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$ $= 4\sqrt{3}$

4	Step 1. Compute the first derivative y. $y = 3 \sin(3x) + \cos(3x).$ Differentiate term by term, using $(\frac{d}{dx} \sin(3x) = 3 \cos(3x))$ and $(\frac{d}{dx} \cos(3x) = -3 \sin(3x))$: $\begin{aligned} \frac{dy}{dx} &= 3[3 \cos(3x)] + 1[-3 \sin(3x)] \\ &= 9 \cos(3x) - 3 \sin(3x) \end{aligned}$ Step 2. Compute the second derivative (y''). Differentiate ($y' = 9 \cos(3x) - 3 \sin(3x)$) again: $\begin{aligned} \frac{d^2y}{dx^2} &= 9[-3 \sin(3x)] - 3[3 \cos(3x)] \\ &= -27 \sin(3x) - 9 \cos(3x) \end{aligned}$ Step 3. Factor and compare with y. Notice $\begin{aligned} &-27 \sin(3x) - 9 \cos(3x) \\ &= -9(3 \sin(3x) + \cos(3x)) \\ &= -9y. \end{aligned}$ Hence, $\frac{d^2y}{dx^2} = -9y \text{ as required.}$
5	1. Identify u and v We view the expression as a product $\begin{aligned} u(x) &= \sin(3x), \\ v(x) &= \tan^2 x + 1. \end{aligned}$ 2. Compute ($u'(x)$) $\begin{aligned} u(x) &= \sin(3x) \\ &\Rightarrow \\ u'(x) &= 3 \cos(3x) \end{aligned}$

	$= \cos(3x) \frac{d}{dx}(3x)$ $= 3\cos(3x).$ 3. Compute $(v'(x))$ $v(x) = \tan^2 x + 1$ $\Rightarrow$ $v'(x)$ $= \frac{d}{dx} [\tan^2 x] + \frac{d}{dx} [1]$ $= 2 \tan x \times \sec^2 x + 0$ $= 2 \tan x \sec^2 x.$ 4. Apply the product rule The product rule says $\frac{d}{dx} [u v]$ $= u' v + u v'.$ Substitute (u, u', v, v'): $\frac{d}{dx} [\sin(3x) (\tan^2 x + 1)]$ $= (3\cos(3x)) (\tan^2 x + 1)$ $+ \sin(3x) (2 \tan x \sec^2 x).$ 5. Write the final answer (no further simplification required) $3\cos(3x) (\tan^2 x + 1) + 2\sin(3x) \tan x \sec^2 x.$
6	$y = \sqrt{\frac{1-x}{x^2+3}}$ $= \frac{\sqrt{1-x}}{\sqrt{x^2+3}}$

Compute $\frac{dy}{dx}$ using the quotient and chain rules

Let

$$u = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$$

$$v = \sqrt{x^2+3} = (x^2+3)^{\frac{1}{2}}$$

so $(y = \frac{u}{v})$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \end{aligned}$$

We have

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2}(1-x)^{-\frac{1}{2}} \times (-1) \\ &= -\frac{1}{2\sqrt{1-x}}, \end{aligned}$$

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times (2x) \\ &= \frac{x}{\sqrt{x^2+3}}. \end{aligned}$$

Hence

$$\left[\frac{dy}{dx} = \frac{\sqrt{x^2+3} \times \left[\frac{1}{2}(1-x)^{-\frac{1}{2}} \times (-1) \right] - \sqrt{1-x} \times \left[\frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times (2x) \right]}{(\sqrt{x^2+3})^2} \right]$$

Step 3. Combine into a single fraction

Put everything over the common denominator $(2(x^2+3)\sqrt{1-x}\sqrt{x^2+3})$:

	$\left[\frac{\sqrt{x^2 + 3} \times \frac{1}{2}(1-x)^{-\frac{1}{2}} + \sqrt{1-x} \times \frac{x}{(x^2 + 3)^{\frac{1}{2}}}}{x^2 + 3} \right]$ $\left[\frac{\frac{\sqrt{x^2 + 3}}{\sqrt{1-x}} \times x - \frac{x\sqrt{1-x}}{\sqrt{x^2 + 3}}}{x^2 + 3} \right]$ $\left[\frac{x(x^2 + 3)\sqrt{x^2 + 3} - x(1-x)\sqrt{1-x}}{\sqrt{1-x} \times \sqrt{x^2 + 3} \times (x^2 + 3)} \right]$ $\left[\frac{dy}{dx} = \frac{x^2 - 2x - 3}{2(x^2 + 3)\sqrt{1-x}\sqrt{x^2 + 3}} \right]$ $\left[\frac{x^2 - 2x - 3}{2(x^2 + 3)\sqrt{1-x}\sqrt{x^2 + 3}} = 0 \right]$ $[x^2 - 2x - 3 = 0]$ $[(x - 3)(x + 1) = 0]$ $[x = 3 \text{ or } x = -1]$
7	1. Identify u and v Let $u(x) = \ln x$, $v(x) = 3x$. 2. Compute (u') and (v') - Since $(u(x) = \ln x)$, $u'(x) = \frac{1}{x}$ - Since $(v(x) = 3x)$, $v'(x) = 3$. 3. Apply the Quotient Rule

	The quotient rule says $\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{u'v - uv'}{v^2}.$ So here $\frac{d}{dx} \left(\frac{\ln x}{3x} \right) = \frac{\left(\frac{1}{x} \right) (3x) - (\ln x) (3)}{(3x)^2}.$ 4. Simplify the numerator Compute the numerator term by term: $\left(\frac{1}{x} \right) (3x) = 3,$ $(\ln x) (3) = 3 \ln x.$ Hence the numerator is $3 - 3 \ln x = 3(1 - \ln x).$ 5. Simplify the denominator $(3x)^2 = 9x^2.$ 6. Put it all together $\frac{d}{dx} \left(\frac{\ln x}{3x} \right) = \frac{3(1 - \ln x)}{9x^2} = \frac{1 - \ln x}{3x^2}.$
8	1. Compute $(h(x))$ in a simpler form. Using the logarithm power-rule, $h(x) = 5 \ln(3x + 4).$ 2. First derivative $(h'(x)).$ Differentiate $(h(x) = 5 \ln(3x + 4))$ with respect to x:

	$h'(x)$ $= 5 \times \frac{d}{dx} [\ln(3x + 4)]$ $= 5 \times \frac{1}{3x + 4} \times 3$ $= \frac{15}{3x + 4}$ 3. Second derivative ($h''(x)$). Differentiate ($h'(x) = 15 \div (3x + 4)$) again: $h''(x)$ $= 15 \times \frac{d}{dx} [(3x + 4)^{-1}]$ $= 15 \times (-1) (3x + 4)^{-2} \times 3$ $= -\frac{45}{(3x + 4)^2}$ 4. Form the combination ($3 h'(x) + (3x + 4) h''(x)$). Substitute the derivatives: $3 h'(x)$ $= 3 \times \frac{15}{3x + 4}$ $= \frac{45}{3x + 4}$ $(3x + 4) h''(x)$ $= (3x + 4) \times \left(-\frac{45}{(3x + 4)^2} \right)$ $= -\frac{45}{3x + 4}$ Hence $3 h'(x) + (3x + 4) h''(x)$ $= \frac{45}{3x + 4} - \frac{45}{3x + 4}$ $= 0.$
9	1. Relate the normal's slope to the given line 1. The given line ($4y - x = 20$) can be written in slope-intercept form: $4y = x + 20$ $\Rightarrow$

$$y = \frac{1}{4}x + 5.$$

Its slope is ($m_{\text{line}} = \frac{1}{4}$).

2. The normal to the curve at Q is parallel to this line, so the normal's slope is also

$$m_{\text{normal}} = \frac{1}{4}.$$

2. Find a and b

1. Compute y' , the derivative of $y' = -\frac{a}{x^2} + b$

$$y' = -\frac{a}{x^2} + b.$$

2. At $(Q(1, -1))$, the slope of the tangent is

$$y'(1) = -a + b.$$

Hence the slope of the normal there is

$$\begin{aligned}
 m_{\text{normal}} &= -\frac{1}{y'(1)} \\
 &= -\frac{1}{-a + b} \\
 &= \frac{1}{a - b}.
 \end{aligned}$$

But we know ($m_{\text{normal}} = \frac{1}{4}$). So

$$\begin{aligned}
 \frac{1}{a - b} &= \frac{1}{4} \\
 \Rightarrow \\
 a - b &= 4.
 \end{aligned}$$

3. Also $Q(1, -1)$ lies on the curve, so

$$\begin{aligned}
 -1 &= \frac{a}{1} + b \times 1 - 1 \\
 \Rightarrow \\
 -1 &= a + b - 1
 \end{aligned}$$

$$\Rightarrow$$

$$a + b = 0.$$

4. Solve the two linear equations

$$\begin{aligned} a - b &= 4 \\ a + b &= 0 \end{aligned}$$

Adding gives ($2a = 4$), so ($a = 2$). Then ($b = -a = -2$).

Answer (i): $a = 2, b = -2$

3. Find the second intersection P of the normal with the curve

1. With ($a = 2, b = -2$), the curve is

$$y = \frac{2}{x} - 2x - 1.$$

2. The normal at Q has equation (point-slope form, slope ($= \frac{1}{4}$)):

$$\begin{aligned} y + 1 &= \frac{1}{4}(x - 1) \\ \Rightarrow \\ y &= \frac{1}{4}x - \frac{5}{4} \end{aligned}$$

3. To find where this meets the curve again, solve simultaneously

$$\frac{2}{x} - 2x - 1 = \frac{1}{4}x - \frac{5}{4}.$$

Multiply both sides by ($4x$):

$$\begin{aligned} 4x \left(\frac{2}{x} - 2x - 1 \right) \\ = 4x \left(\frac{1}{4}x - \frac{5}{4} \right), \end{aligned}$$

$$\begin{aligned} 8 - 8x^2 - 4x &= x^2 - 5x \\ \Rightarrow \\ -9x^2 + x + 8 &= 0 \\ \Rightarrow \\ 9x^2 - x - 8 &= 0. \end{aligned}$$

	The quadratic factors or use the formula: $x = \frac{1 \pm 17}{18}$ 1 or $-\frac{8}{9}$. $(x = 1)$ gives the known point (Q). The other intersection is $(x = -\frac{8}{9})$. 4. Substitute $(x = -\frac{8}{9})$ into the normal's equation to get (y): $ \begin{aligned} y &= \frac{1}{4} \left(-\frac{8}{9} \right) - \frac{5}{4} \\ &= -\frac{2}{9} - \frac{5}{4} \\ &= -\frac{8}{36} - \frac{45}{36} \\ &= -\frac{53}{36}. \end{aligned} $ Answer (ii): $P = \left(-\frac{8}{9}, -\frac{53}{36} \right).$
10	(a) Let $u = \frac{\pi}{3} + 3x,$ so the function is $(f(x) = [\tan(u)]^3)$. Then by the chain rule: - Outer derivative: $\frac{d}{du}(\tan^3 u) = 3\tan^2 u \times \sec^2 u$ - Inner derivative: $\frac{du}{dx} = 3$ Hence $ \begin{aligned} &\frac{d}{dx} \tan^3 \left(\frac{\pi}{3} + 3x \right) \\ &= 3 \tan^2(u) \sec^2(u) \times 3 \\ &= 9 \tan^2 \left(\frac{\pi}{3} + 3x \right) \sec^2 \left(\frac{\pi}{3} + 3x \right). \end{aligned} $ (b) We have

$$y = 4 \cos(3x) + \sin x.$$

Differentiate term by term:

$$\begin{aligned} & -\left(\frac{d}{dx} [4 \cos(3x)]\right) \\ &= 4 \times (-\sin(3x)) \times 3 \\ &= -12 \sin(3x) \end{aligned}$$

$$-\left(\frac{d}{dx} [\sin x]\right) = \cos x$$

Therefore

$$y' = -12 \sin(3x) + \cos x.$$

At $(x = \frac{\pi}{6})$, we compute

$$\begin{aligned} & \sin\left(3 \times \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{2}\right) \\ &= 1, \end{aligned}$$

$$\begin{aligned} & \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Hence

$$\begin{aligned} & y'\left(\frac{\pi}{6}\right) \\ &= -12 \times 1 + \frac{\sqrt{3}}{2} \\ &= -12 + \frac{\sqrt{3}}{2} \\ &= \frac{-24 + \sqrt{3}}{2}. \end{aligned}$$

So the exact gradient is

$$\boxed{-12 + \frac{\sqrt{3}}{2}}$$

	or $\frac{-24 + \sqrt{3}}{2}$
11	1. Locate the x-intercept. On the curve, $y=0$ if and only if $(ax - 2) \ln x = 0.$ Since $(\ln x = 0)$ gives $(x = 1)$ (and we exclude the other root $(x = 2/a)$), the curve crosses the x-axis at $P = (1,0)$. 2. Compute the derivative. $y = (ax - 2) \ln x$ $\Rightarrow$ $y' = (ax - 2) \frac{1}{x} + a \ln x$ $= a - \frac{2}{x} + a \ln x.$ At $(x = 1)$, $\ln 1 = 0$, so $y'(1) = a - 2.$ Thus the slope of the tangent at P is $(m_{\text{tan}} = a - 2)$. 3. Find the slope of the normal. The normal slope is the negative reciprocal of the tangent slope: $m_{\text{norm}} = -\frac{1}{m_{\text{tan}}}$ $= -\frac{1}{a - 2}.$ 4. Enforce parallelism condition. The line $(x + 4y - 3 = 0)$ can be written $(y = -\frac{1}{4}x + \frac{3}{4})$, so its slope is $(-\frac{1}{4})$. We require $m_{\text{norm}} = -\frac{1}{4}$ $\Rightarrow$

	$-\frac{1}{a-2} = -\frac{1}{4}$ $\Rightarrow$ $\frac{1}{a-2} = \frac{1}{4}$ $\Rightarrow$ $a-2 = 4$ $\Rightarrow$ $a = 6.$
12	1. First compute the gradient (the first derivative): $y' = \frac{d}{dx}(x^4 - 2x^3 - x^2 + x + 5)$ $= 4x^3 - 6x^2 - 2x + 1.$ 2. The gradient (y') is increasing exactly when its derivative y'' is positive: $y'' = \frac{d}{dx}(4x^3 - 6x^2 - 2x + 1)$ $= 12x^2 - 12x - 2.$ Set ($y'' > 0$): $12x^2 - 12x - 2 > 0$ $\Leftrightarrow$ $6x^2 - 6x - 1 > 0.$ 3. Solve the quadratic inequality ($6x^2 - 6x - 1 = 0$) to find the critical points: $x = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 6 \times (-1)}}{2 \times 6}$ $= \frac{6 \pm \sqrt{36 + 24}}{12}$ $= \frac{6 \pm \sqrt{60}}{12}$ $= \frac{6 \pm 2\sqrt{15}}{12}$ $= \frac{1}{2} \pm \frac{\sqrt{15}}{6}.$ Thus the roots are

	$x_1 = \frac{1}{2} - \frac{\sqrt{15}}{6}$ $x_2 = \frac{1}{2} + \frac{\sqrt{15}}{6}$ 4. A positive-leading quadratic ($6x^2 - 6x - 1$) is positive outside the interval between its roots. Hence answer would be $6x^2 - 6x - 1 > 0$ $\Leftrightarrow$ $x < \frac{1}{2} - \frac{\sqrt{15}}{6}$ or $x > \frac{1}{2} + \frac{\sqrt{15}}{6}$
13	1. Compute the derivative $\frac{dy}{dx}$ We use the quotient rule: if ($y = \frac{u(x)}{v(x)}$), then $\frac{dy}{dx} = \frac{u'v - v'u}{v^2}.$ Here $u(x) = 5x - 2, \quad u'(x) = 5,$ $v(x) = 3x + 1, \quad v'(x) = 3.$ So $\frac{dy}{dx} = \frac{5(3x + 1) - 3(5x - 2)}{(3x + 1)^2}$ $= \frac{15x + 5 - 15x + 6}{(3x + 1)^2}$ $= \frac{11}{(3x + 1)^2}.$ 2. Determine the sign of $\frac{dy}{dx}$

	- The numerator is the constant ($11 > 0$). - The denominator $((3x + 1)^2)$ is a perfect square, hence always positive so long as $(3x + 1 \neq 0)$. Since, $x \neq \frac{1}{3}$, the denominator is indeed (> 0). Therefore $\frac{dy}{dx} = \frac{11}{(3x + 1)^2} > 0$ for all allowed x. 3. Conclusion Because $\frac{dy}{dx} > 0$ everywhere on its domain, the function $(y = \frac{5x-2}{3x+1})$ is strictly increasing as x increases.
14	1. Compute the derivative Differentiate term-by-term: $\begin{aligned} \frac{dy}{dx} &= -\frac{1}{4} \times 3x^2 + 2x - 5 \\ &= -\frac{3}{4}x^2 + 2x - 5. \end{aligned}$ 2. Factor out the negative coefficient $\frac{dy}{dx} = -\frac{3}{4} \left(x^2 - \frac{8}{3}x + \frac{20}{3} \right)$ 3. Complete the square inside the parentheses Inside the big parentheses consider $\begin{aligned} x^2 - \frac{8}{3}x &= \left(x - \frac{4}{3} \right)^2 - \left(\frac{4}{3} \right)^2 \\ &= \left(x - \frac{4}{3} \right)^2 - \frac{16}{9}. \end{aligned}$ Hence $x^2 - \frac{8}{3}x + \frac{20}{3}$

	$= \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + \frac{20}{3}$ $= \left(x - \frac{4}{3}\right)^2 + \frac{44}{9}.$ 4. Substitute back $\frac{dy}{dx} = -\frac{3}{4} \left[\left(x - \frac{4}{3}\right)^2 + \frac{44}{9} \right].$ 5. Conclude monotonicity - For all real (x), $\left(x - \frac{4}{3}\right)^2 \geq 0$, so $\left(x - \frac{4}{3}\right)^2 + \frac{44}{9} > 0.$ - Multiplying a strictly positive quantity by $\left(-\frac{3}{4}\right)$ gives a strictly negative result. Therefore $\frac{dy}{dx} < 0$ for all real x, which means (y) is strictly decreasing on (R).
15	(i) Differentiating y We use the quotient rule. If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}.$ Here $u(x) = 4x^2, \quad u'(x) = 8x,$ $v(x) = 3 - 2x, \quad v'(x) = -2.$ So $\frac{dy}{dx}$

$$\begin{aligned}
 &= \frac{(8x)(3-2x) - (4x^2)(-2)}{(3-2x)^2} \\
 &= \frac{24x - 16x^2 + 8x^2}{(3-2x)^2} \\
 &= \frac{24x - 8x^2}{(3-2x)^2} \\
 &= \frac{8x(3-x)}{(3-2x)^2}
 \end{aligned}$$

Answer to (i): $\frac{dy}{dx} = \frac{8x(3-x)}{(3-2x)^2}$.

(ii) Finding $\left(\frac{dy}{dt}\right)$ when $x=4$

We are given $\left(\frac{dx}{dt} = 0.06\right) \text{ units/s}$. By the chain rule,

$$\begin{aligned}
 &\frac{dy}{dt} \\
 &= \frac{dy}{dx} \frac{dx}{dt}.
 \end{aligned}$$

First evaluate $\left(\frac{dy}{dx}\right)$ at $x = 4$:

$$\begin{aligned}
 &\left.\frac{dy}{dx}\right|_{x=4} \\
 &= \frac{8 \times 4 (3-4)}{(3-2 \times 4)^2} \\
 &= \frac{32 \times (-1)}{(3-8)^2} \\
 &= \frac{-32}{(-5)^2} \\
 &= -\frac{32}{25}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\frac{dy}{dt} \\
 &= \left(-\frac{32}{25}\right) (0.06) \\
 &= -\frac{32}{25} \times \frac{6}{100}
 \end{aligned}$$

	$= -\frac{192}{2500}$ $= -0.0768 \text{ units/s.}$ Answer to (ii): $\frac{dy}{dt} = -\frac{32}{25} \times 0.06 = -0.0768 \text{ units/s (approximately).}$
16	(i) Differentiate ($y = \ln(7x \div (5x + 2))$) We use the chain rule and the quotient inside the log. 1. Write $y = \ln(u(x))$ $u(x) = \frac{7x}{5x + 2}.$ 2. Then $\frac{dy}{dx} = \frac{1}{u(x)} \times u'(x).$ 3. Now compute $u'(x) = \frac{(7)(5x + 2) - (7x)(5)}{(5x + 2)^2}$ $= \frac{35x + 14 - 35x}{(5x + 2)^2}$ $= \frac{14}{(5x + 2)^2}$ 4. Hence $\frac{dy}{dx} = \frac{1}{\frac{7x}{5x + 2}} \times \frac{14}{(5x + 2)^2}$ $= \frac{5x + 2}{7x} \times \frac{14}{(5x + 2)^2}$

$$\begin{aligned}
 &= \frac{14}{7x(5x+2)} \\
 &= \frac{2}{x(5x+2)}
 \end{aligned}$$

So the answer for (i) is

$$\frac{dy}{dx} = \frac{2}{x(5x+2)}.$$

(ii) Related rates when the curve crosses the x-axis

1. Find the crossing point: On the (x)-axis, (y = 0). So

$$\begin{aligned}
 0 &= \ln\left(\frac{7x}{5x+2}\right) \\
 \Rightarrow \\
 \frac{7x}{5x+2} &= 1 \\
 \Rightarrow \\
 7x &= 5x + 2 \\
 \Rightarrow \\
 2x &= 2 \\
 \Rightarrow \\
 x &= 1.
 \end{aligned}$$

2. We know $\left(\frac{dy}{dt} = 0.4\right)$. We seek $\left(\frac{dx}{dt}\right)$ at $(x = 1)$.

By the chain rule,

$$\begin{aligned}
 \frac{dy}{dt} \\
 = \frac{dy}{dx} \times \frac{dx}{dt}.
 \end{aligned}$$

So

$$\begin{aligned}
 0.4 &= \frac{2}{x(5x+2)} \\
 &\times \frac{dx}{dt}.
 \end{aligned}$$

3. Evaluate $\left(\frac{dy}{dx}\right)$ at $(x = 1)$ frac:

$$\begin{aligned}
 \frac{dy}{dx} \Big|_{x=1} \\
 = \frac{2}{1(5 \times 1 + 2)}
 \end{aligned}$$

	$= \frac{2}{7}.$ 4. Solve for $\left(\frac{dx}{dt}\right)$: $0.4 = \frac{2}{7} \frac{dx}{dt}$ $\Rightarrow$ $\frac{dx}{dt}$ $= 0.4 \times \frac{7}{2}$ $= 0.4 \times 3.5$ $= 1.4.$ Thus, when the curve crosses the x-axis, x is increasing at $\frac{dx}{dt} = 1.4 \text{ units/sec.}$
17	Step 1. Differentiate both sides with respect to time (t). Write $(y = 8x^{-2} + 3x^{-1})$. $\frac{dy}{dt}$ $= \frac{d}{dt}(8x^{-2} + 3x^{-1})$ $= 8 \times (-2)x^{-3} \frac{dx}{dt} + 3 \times (-1)x^{-2} \frac{dx}{dt}$ $= (-16x^{-3} - 3x^{-2}) \frac{dx}{dt}.$ So $\frac{dy}{dt}$ $= \left(-\frac{16}{x^3} - \frac{3}{x^2}\right) \frac{dx}{dt}.$ Step 2. Find the value of (x) when $(y = 3.5)$. Solve $3.5 = \frac{8}{x^2} + \frac{3}{x}.$ Multiply through by (x^2): $3.5x^2 = 8 + 3x$

	$\Rightarrow 3.5x^2 - 3x - 8 = 0.$ Multiply by 2 to clear the decimal: $7x^2 - 6x - 16 = 0.$ Use the quadratic formula: $x = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 7 \times (-16)}}{2 \times 7}$ $= \frac{6 \pm \sqrt{36 + 448}}{14}$ $= \frac{6 \pm \sqrt{484}}{14}$ $= \frac{6 \pm 22}{14}.$ Thus $(x = 2)$ or $(x = -\frac{8}{7})$. We take the positive root $(x = 2)$. Step 3. Substitute $(x = 2)$ and the known $(\frac{dy}{dt} = 1.6)$ into the differentiated formula: $1.6 = \left(-\frac{16}{2^3} - \frac{3}{2^2}\right) \frac{dx}{dt}$ $= (-2 - 0.75) \frac{dx}{dt}$ $= -2.75 \frac{dx}{dt}.$ Therefore $\frac{dx}{dt} = \frac{1.6}{-2.75}$ $\approx -0.5818 \text{ units/s.}$ Answer: $(\frac{dx}{dt} \approx -0.582) \text{ units s}^{-1}$.
18	1. Compute the first derivative $f(x) = 2x^3 - 3x^2 - 12x + 6$ $\Rightarrow$ $f'(x) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(3x^2) - \frac{d}{dx}(12x) + \frac{d}{dx}(6)$ $= 6x^2 - 6x - 12.$

Factorize:

$$f'(x) = 6(x^2 - x - 2)$$

$$= 6(x - 2)(x + 1).$$

2. Find critical points

Set $(f'(x) = 0)$:

$$6(x - 2)(x + 1) = 0$$

$$\Rightarrow$$

$$x = 2, -1.$$

3. Compute the corresponding (y) -values

1. At $(x = -1)$:

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 6$$

$$= -2 - 3 + 12 + 6 = 13.$$

Turning point: $((-1, 13).)$

2. At $(x = 2)$:

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 6$$

$$= 16 - 12 - 24 + 6 = -14.$$

Turning point: $((2, -14).)$

4. Determine nature via the second derivative

$$f''(x) = \frac{d}{dx}(6x^2 - 6x - 12)$$

$$= 12x - 6.$$

- At $(x = -1)$:

$$(f''(-1) = 12(-1) - 6 = -18 < 0), \text{ so } (f) \text{ has a local maximum at } ((-1, 13)).$$

- At $(x = 2)$:

$$(f''(2) = 12 \times 2 - 6 = 18 > 0), \text{ so } (f) \text{ has a local minimum at } ((2, -14)).$$

Answer.

Turning points:

- Local maximum at $((-1, 13)).$

- Local minimum at $((2, -14)).$

19 1. Simplify the function

$$\begin{aligned} y &= \frac{16x^3 + 1}{2x^2} \\ &= \frac{16x^3}{2x^2} + \frac{1}{2x^2} \\ &= 8x + \frac{1}{2}x^{-2}. \end{aligned}$$

2. Compute the first derivative

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(8x) \\ &+ \frac{d}{dx}\left(\frac{1}{2}x^{-2}\right) \\ &= 8 \\ &+ \frac{1}{2} \times (-2)x^{-3} \\ &= 8 - x^{-3}. \end{aligned}$$

3. Find critical points

Set $\left(\frac{dy}{dx} = 0\right)$:

$$\begin{aligned} 8 - x^{-3} &= 0 \\ \Rightarrow \\ x^{-3} &= 8 \\ \Rightarrow \\ x^3 &= \frac{1}{8} \\ \Rightarrow \\ x &= \frac{1}{2}. \end{aligned}$$

4. Compute the corresponding (y) – value

$$\begin{aligned} y\left(\frac{1}{2}\right) &= 8 \times \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^{-2} \\ &= 4 + \frac{1}{2} \times 4 \\ &= 4 + 2 \\ &= 6. \end{aligned}$$

	So the turning point is $(\frac{1}{2}, 6)$. 5. Determine maximum or minimum Compute the second derivative: $\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(8 - x^{-3}) \\ &= 0 + 3x^{-4} \\ &= \frac{3}{x^4}.\end{aligned}$ For $(x = \frac{1}{2})$, $(\frac{3}{(\frac{1}{2})^4} = \frac{3}{\frac{1}{16}} = 48 > 0)$ Since $(\frac{d^2y}{dx^2} > 0)$, the curve is concave up at $(x = \frac{1}{2})$, so the turning point is a minimum. Answer: The curve has a turning point at $(\frac{1}{2}, 6)$, and it is a minimum.
20	(i) Expressing A in terms of (x) 1. Total wire-netting length constraint. - There are two horizontal runs of netting, each of length $(3x)$, so horizontal total is $2 \times (3x) = 6x \text{ metres.}$ - There are four vertical runs (the two ends and the two internal dividers), each of length (y), so vertical total is $4y \text{ metres.}$ - The sum is 288 m, so $\begin{aligned}6x + 4y &= 288 \\ \Rightarrow 4y &= 288 - 6x \\ \Rightarrow y &= 72 - 1.5x.\end{aligned}$ 2. Area of three courts. Each court has area $(x \times y)$, so three courts have total area $A = 3xy.$

	Substitute ($y = 72 - 1.5x$): $ \begin{aligned} A &= 3x(72 - 1.5x) \\ &= 216x - 4.5x^2 \\ &= 216x - \frac{9}{2}x^2, \end{aligned} $ as required. (ii) Maximizing ($A(x) = 216x - \frac{9}{2}x^2$) To find the critical point, differentiate w.r.t. x: $\frac{dA}{dx} = 216 - 9x.$ Set ($\frac{dA}{dx} = 0$) to find the stationary point: $ \begin{aligned} 216 - 9x &= 0 \\ \Rightarrow \\ 9x &= 216 \\ \Rightarrow \\ x &= 24 \text{ m.} \end{aligned} $ Then $y = 72 - 1.5(24) = 72 - 36 = 36 \text{ m.}$ Hence each court is 24m by 36m when the total area is maximized. Answer: (i) ($A = 216x - \frac{9}{2}x^2$.) (ii) The maximum occurs at $x=24\text{m}$ and $y=36\text{m}$.
21	1. Compute the derivative ($y'(x)$) and set it to zero to locate stationary points: $ \begin{aligned} y' &= \frac{d}{dx}(x^3 - 3x^2 - 24x + c) \\ &= 3x^2 - 6x - 24. \end{aligned} $ Setting ($y' = 0$) gives $3x^2 - 6x - 24 = 0$

$$\Rightarrow x^2 - 2x - 8 = 0.$$

Factor or use the quadratic formula:

$$x^2 - 2x - 8 = (x - 4)(x + 2) = 0$$

$$\Rightarrow x = 4 \quad \text{or} \quad x = -2.$$

Thus the stationary points occur at $(x = -2)$ and $(x = 4)$.

2. Determine which of these is a minimum. Compute the second derivative:

$$y'' = \frac{d}{dx}(3x^2 - 6x - 24) = 6x - 6.$$

Evaluate at each stationary (x) :

- At $(x = -2)$:

$$(y''(-2) = 6(-2) - 6 = -18 < 0) \rightarrow \text{local maximum.}$$

- At $(x = 4)$:

$$(y''(4) = 6 \times 4 - 6 = 18 > 0) \rightarrow \text{local minimum.}$$

We want this minimum point $(4, y(4))$ to lie on the x-axis, so $(y(4) = 0)$. Compute:

$$\begin{aligned}
 y(4) &= 4^3 - 3 \times 4^2 - 24 \times 4 + c \\
 &= 64 - 48 - 96 + c \\
 &= -80 + c.
 \end{aligned}$$

Setting $(-80 + c = 0)$ gives

$$c = 80.$$

Answer

(i) The stationary points are at $(x = -2)$ and $(x = 4)$.

(ii) The value of c that makes the minimum lie on the x-axis is 80.