

# 7 ESSENTIAL QUESTIONS TO MASTER

## POLYNOMIALS

|   |                                                                                                                                                                                                                                                                                                                                                                                                                                      |  |
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| 1 | (a) Multiply $(12x^2 - 2x + 5)$ by $(2x + 3)$ .<br>(b) Find the coefficient of $x^2$ in the product of $(x^2 + x - 2)$ and $(2x^2 + 5x - 12)$ .                                                                                                                                                                                                                                                                                      |  |
| 2 | Factorise completely the cubic polynomial<br>$2x^3 - 3x^2 + 1$ .                                                                                                                                                                                                                                                                                                                                                                     |  |
| 3 | Given that $4x^2 - 12x + 7 \equiv a(x + b)^2 + c$ , find the values of a, b and c.                                                                                                                                                                                                                                                                                                                                                   |  |
| 4 | Given that the cubic<br>$f(x) = px^3 + 5x^2 + qx - 6$ is exactly divisible by the quadratic<br>$2x^2 + x - 6$<br>find the values of p and q.                                                                                                                                                                                                                                                                                         |  |
| 5 | Given $(f(x) = 2x^3 - 5x^2 - 4x + 12)$ ,<br><br>(a) Show that $((x - 2))$ is a factor of $(f(x))$ .<br><br>(b) Factorise $(f(x))$ completely.<br><br>(c) Hence, solve the equation<br>$2(2^{3y}) - 5(2^{2y}) = 4(2^{y-3}).$                                                                                                                                                                                                          |  |
| 6 | The polynomial $(f(x))$ is defined by<br><br>$f(x) = 3x^3 - 4x^2 + qx + 6$ ,<br><br>where $(q)$ is a constant. When $(f(x))$ is divided by $(x - 1)$ , the remainder is $(-12)$ .<br><br>(a) Show that $(q = -17)$ .<br>(b) Hence solve the equation $(f(x) = 0)$ .<br>(c) Use your answer to part (b) to solve the equation<br>$3(4^x)^3 - 4(4^x)^2 - 17(4^x) + 6 = 0.$<br>(d) Explain why there are only two solutions for $(x)$ . |  |
| 7 | (a) Show that $(x + 1)$ is a factor of<br>$2x^3 + 11x^2 + 22x + 13$ .<br><br>(b) Jacob said that $(y = 2x^3 + 11x^2 + 22x + 13)$ is a cubic graph, therefore the graph will cut the $(x)$ -axis at 3 points. Do you agree? Explain clearly by showing all your working.                                                                                                                                                              |  |

**Answer Key:**

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| 1 | <p><b>Solution:</b></p> <p>(a) We compute <math>((12x^2 - 2x + 5)(2x + 3))</math> by distribution:<br/> <math>(12x^2)(2x + 3) = 24x^3 + 36x^2</math><br/> <math>(-2x)(2x + 3) = -4x^2 - 6x</math><br/> <math>(5)(2x + 3) = 10x + 15</math>.<br/>         Summing,<br/> <math>24x^3 + (36x^2 - 4x^2) + (-6x + 10x) + 15</math><br/> <math>= 24x^3 + 32x^2 + 4x + 15</math>.</p> <p>Answer (a): <math>24x^3 + 32x^2 + 4x + 15</math></p> <p>(b) Let <math>(A = x^2 + x - 2)</math> and <math>(B = 2x^2 + 5x - 12)</math> We want the coefficient of <math>(x^2)</math> in <math>(A \times B)</math>.</p> <p>When multiplying two polynomials, the <math>(x^2)</math>-terms arise from:</p> <ul style="list-style-type: none"> <li>- <math>((x^2) \times (\text{constant term of } B))</math>, which is <math>(x^2 \times (-12) = -12x^2)</math>.</li> <li>- <math>((x) \times (5x))</math>, which is <math>(5x^2)</math>.</li> <li>- <math>((\text{constant term of } A) \times (2x^2))</math>, which is <math>((-2) \times 2x^2 = -4x^2)</math>.</li> </ul> <p>No other pair of terms gives exactly <math>(x^2)</math>. Adding these contributions,<br/> <math>-12 + 5 + (-4) = -11</math>.</p> <p>Answer (b): The coefficient of <math>(x^2)</math> is <math>(-11)</math>.</p> |
| 2 | <p>1. Check for integer (or simple rational) roots by the Rational Root Test. Possible roots are <math>(\pm 1, \pm \frac{1}{2})</math>.</p> <p>2. Evaluate at <math>(x = 1)</math>:<br/> <math>(2(1)^3 - 3(1)^2 + 1 = 2 - 3 + 1 = 0.)</math><br/>         So <math>(x = 1)</math> is a root, and <math>((x - 1))</math> is a factor.</p> <p>3. Perform polynomial (or synthetic) division of <math>(2x^3 - 3x^2 + 0x + 1)</math> by <math>((x - 1))</math>:</p> <p>Dividing yields the quotient <math>(2x^2 - x - 1)</math>.</p> <p>4. Factor the quadratic:<br/> <math>2x^2 - x - 1 = (2x + 1)(x - 1)</math>.</p> <p>5. Therefore, the complete factorisation is</p> $  \begin{aligned}  &2x^3 - 3x^2 + 1 \\  &= (x - 1)(2x^2 - x - 1) \\  &= (x - 1)(2x + 1)(x - 1) \\  &= (x - 1)^2(2x + 1).  \end{aligned}  $                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| 3 | <p><b>Solution (by completing the square):</b></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |

1. Start with

$$4x^2 - 12x + 7.$$

2. Factor out the 4 from the  $(x)$  – terms:

$$4(x^2 - 3x) + 7.$$

3. Complete the square inside the parentheses. We want

$$\begin{aligned}
 x^2 - 3x &= \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 \\
 &= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}.
 \end{aligned}$$

So

$$4(x^2 - 3x) + 7$$

$$\begin{aligned}
 &= 4 \left[ \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} \right] + 7 \\
 &= 4 \left(x - \frac{3}{2}\right)^2 - 4 \times \frac{9}{4} + 7 \\
 &= 4 \left(x - \frac{3}{2}\right)^2 - 9 + 7 \\
 &= 4 \left(x - \frac{3}{2}\right)^2 - 2.
 \end{aligned}$$

4. Comparing

$$\begin{aligned}
 &4 \left(x - \frac{3}{2}\right)^2 - 2 \\
 &\quad \text{with} \\
 &a(x + b)^2 + c
 \end{aligned}$$

we read off

$$\begin{aligned}
 a &= 4, \\
 x + b &= x - \frac{3}{2} \Rightarrow b = -\frac{3}{2}, \\
 c &= -2.
 \end{aligned}$$

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|   | <p>Answer:</p> $(a = 4, \quad b = -\frac{3}{2}, \quad c = -2.)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| 4 | <p>Solution.</p> <p>1. Factor the divisor:</p> $2x^2 + x - 6 = (x + 2)(2x - 3).$ <p>Hence <math>(x + 2)</math> and <math>(2x - 3)</math> must each be factors of <math>(f(x))</math>.</p> <p>2. Since <math>(x = -2)</math> is a root of the divisor, we must have</p> $f(-2) = 0.$ <p>Compute—</p> $\begin{aligned} f(-2) &= p(-2)^3 + 5(-2)^2 + q(-2) - 6 \\ &= -8p + 20 - 2q - 6 \\ &= -8p - 2q + 14 = 0. \end{aligned}$ <p>Rearranging gives</p> $\begin{aligned} 8p + 2q &= 14 \\ \Rightarrow \\ 4p + q &= 7 \\ (\text{Equation 1}). \end{aligned}$ <p>3. Since <math>x = \frac{3}{2}</math> is the other root of the divisor, we must have</p> $f\left(\frac{3}{2}\right) = 0.$ <p>Compute</p> $\begin{aligned} f\left(\frac{3}{2}\right) &= p\left(\frac{3}{2}\right)^3 \\ &+ 5\left(\frac{3}{2}\right)^2 \\ &+ q\left(\frac{3}{2}\right) \\ &- 6 \end{aligned}$ |

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|   | $= \frac{27}{8}p + \frac{45}{4} + \frac{3}{2}q - 6 = 0.$ <p>Rewrite (<math>\frac{45}{4} - 6 = \frac{45-24}{4} = \frac{21}{4}</math>), so</p> $\frac{27}{8}p + \frac{3}{2}q + \frac{21}{4} = 0.$ <p>Multiply through by 8 to clear denominators:</p> $27p + 12q + 42 = 0$ $\Rightarrow$ $27p + 12q = -42$ $\Rightarrow$ $9p + 4q = -14$ <p>(Equation 2).</p> <p>4. Solve the system</p> $4p + q = 7$ $9p + 4q = -14$                                                                                                                                                                                                                                                                                                                                                                 |
| 5 | <p>(a) Show <math>((x - 2))</math> is a factor</p> <p>By the Factor Theorem, <math>((x - 2))</math> is a factor of <math>(f(x))</math> exactly when <math>(f(2) = 0)</math>.</p> $f(2) = 2 \times 2^3 - 5 \times 2^2 - 4 \times 2 + 12$ $= 2 \times 8 - 5 \times 4 - 8 + 12$ $= 16 - 20 - 8 + 12$ $= 0.$ <p>Since <math>(f(2) = 0)</math>, <math>((x - 2))</math> is indeed a factor of <math>(f(x))</math>.</p> <p>(b) Factorise <math>(f(x))</math> completely</p> <p>We divide <math>(f(x))</math> by <math>((x - 2))</math>. One can do polynomial long-division or synthetic division; the result is</p> $\begin{array}{r} 2x^3 - 5x^2 - 4x + 12 \\ x - 2 \\ \hline 2x^2 - x - 6. \end{array}$ <p>Now factor the quadratic:</p> $2x^2 - x - 6 = (2x + 3)(x - 2).$ <p>Hence</p> |

$$\begin{aligned}
 f(x) &= 2x^3 - 5x^2 - 4x + 12 \\
 &= (x-2)(2x^2 - x - 6) \\
 &= (x-2)(2x+3)(x-2) \\
 &= (x-2)^2(2x+3).
 \end{aligned}$$

(c) Solve  $(2(2^{3y}) - 5(2^{2y}) = 4(2^{y-3}))$

1. Let  $(t = 2^y)$ . Then

$$\begin{aligned}
 2^{3y} &= (2^y)^3 = t^3, \\
 2^{2y} &= t^2, \\
 2^{y-3} &= 2^y \times 2^{-3} = t \times \frac{1}{8} = \frac{t}{8}.
 \end{aligned}$$

2. Substitute into the equation:

$$\begin{aligned}
 2t^3 - 5t^2 &= 4\left(\frac{t}{8}\right) \\
 \Rightarrow \\
 2t^3 - 5t^2 &= \frac{t}{2}.
 \end{aligned}$$

3. Multiply both sides by 2:

$$\begin{aligned}
 4t^3 - 10t^2 &= t \\
 \Rightarrow \\
 4t^3 - 10t^2 - t &= 0 \\
 \Rightarrow \\
 t(4t^2 - 10t - 1) &= 0.
 \end{aligned}$$

Since  $(t = 2^y > 0)$ , the factor  $t=0$  is extraneous. Solve

$$\begin{aligned}
 4t^2 - 10t - 1 &= 0 \\
 \Rightarrow \\
 t &= \frac{10 \pm \sqrt{100 + 16}}{8} \\
 &= \frac{10 \pm \sqrt{116}}{8} \\
 &= \frac{10 \pm 2\sqrt{29}}{8} \\
 &= \frac{5 \pm \sqrt{29}}{4}.
 \end{aligned}$$

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|   | <p>Of these, <math>(\frac{5-\sqrt{29}}{4} &lt; 0)</math> is not admissible (since <math>t &gt; 0</math>), so</p> $t = \frac{5+\sqrt{29}}{4}.$ <p>4. Finally,</p> $2^y = t = \frac{5+\sqrt{29}}{4}$ $\Rightarrow$ $y = \log_2\left(\frac{5+\sqrt{29}}{4}\right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| 6 | <p>(a) By the Remainder Theorem, the remainder when dividing by <math>(x - 1)</math> is <math>(f(1))</math>. We are told <math>f(1) = -12</math>.</p> <p>But</p> $f(1) = 3(1)^3 - 4(1)^2 + q(1) + 6 = 3 - 4 + q + 6 = 5 + q.$ <p>Thus</p> $5 + q = -12$ $\Rightarrow$ $q = -12 - 5 = -17.$ <p>From now on,</p> $f(x) = 3x^3 - 4x^2 - 17x + 6.$ <p>(a) (<math>q = -17</math>).</p> <p>(b) We solve <math>(3x^3 - 4x^2 - 17x + 6 = 0)</math>. First look for an integer (or simple rational) root by inspection. Test <math>(x = 3)</math>:</p> $f(3) = 3 \times 27 - 4 \times 9 - 17 \times 3 + 6$ $= 81 - 36 - 51 + 6 = 0.$ <p>So <math>(x = 3)</math> is a root, and hence <math>((x - 3))</math> is a factor. Perform polynomial (or synthetic) division of <math>(3x^3 - 4x^2 - 17x + 6)</math> by <math>((x - 3))</math>:</p> <p>Division yields</p> |

$$3x^3 - 4x^2 - 17x + 6 = (x - 3)(3x^2 + 5x - 2).$$

Next solve the quadratic

$$\begin{aligned} 3x^2 + 5x - 2 &= 0 \\ \Rightarrow \\ x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\ &= \frac{-5 \pm \sqrt{25 + 24}}{6} \\ &= \frac{-5 \pm 7}{6} \end{aligned}$$

Hence

$$x = \frac{-5 + 7}{6} = \frac{2}{6} = \frac{1}{3},$$

$$x = \frac{-5 - 7}{6} = -\frac{12}{6} = -2.$$

(b) The three solutions of  $(f(x) = 0)$  are

$$x = 3, \quad x = \frac{1}{3}, \quad x = -2.$$

(c) Let  $(t = 4^x)$ . Then the equation

$$3(4^x)^3 - 4(4^x)^2 - 17(4^x) + 6 = 0$$

becomes

$$3t^3 - 4t^2 - 17t + 6 = 0.$$

But this is exactly  $(f(t) = 0)$  with  $(q = -17)$ . From part (b) its solutions are  $(t = 3, t = \frac{1}{3}, t = -2)$ . Since  $(t = 4^x > 0)$ , we discard  $(t = -2)$ . We are left with

$$\begin{aligned} 4^x &= 3 \\ \Rightarrow \\ x &= \frac{\ln 3}{\ln 4}, \end{aligned}$$

and

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|   | $4^x = \frac{1}{3}$ $\Rightarrow$ $x = \frac{\ln(\frac{1}{3})}{\ln 4} = -\frac{\ln 3}{\ln 4}.$ <p>Hence the two real solutions are</p> $x = \frac{\ln 3}{\ln 4} \text{ or } x = -\frac{\ln 3}{\ln 4}.$ <p>(d) There are only two solutions because (<math>4^x &gt; 0</math>) rules out the negative root (<math>t = -2</math>) of the cubic in part (c), leaving just the two positive values (<math>t = 3</math>) and (<math>t = \frac{1}{3}</math>).</p>                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| 7 | <p>(a) Let <math>(f(x) = 2x^3 + 11x^2 + 22x + 13)</math>.<br/>Test (<math>x = -1</math>):</p> $  \begin{aligned}  f(-1) &= 2(-1)^3 + 11(-1)^2 + 22(-1) + 13 \\  &= 2(-1) + 11(1) - 22 + 13 \\  &= -2 + 11 - 22 + 13 \\  &= 0.  \end{aligned}  $ <p>Therefore <math>(x + 1)</math> is a factor of <math>(f(x))</math>.</p> <p>(b) Divide <math>(2x^3 + 11x^2 + 22x + 13)</math> by <math>(x + 1)</math> (long division, coefficient comparison, or synthetic division) to get:</p> $2x^3 + 11x^2 + 22x + 13 = (x + 1)(2x^2 + 9x + 13).$ <p>To find the x-intercepts solve<br/> <math>(x + 1)(2x^2 + 9x + 13) = 0</math>.</p> <p>This gives</p> $  \begin{aligned}  x &= -1 \\  \text{or} \\  2x^2 + 9x + 13 &= 0.  \end{aligned}  $ <p>For the quadratic <math>(2x^2 + 9x + 13)</math>, compute the discriminant:</p> $\Delta = b^2 - 4ac = 9^2 - 4 \cdot 2 \cdot 13 = 81 - 104 = -23 < 0.$ |

Since  $(\Delta < 0)$ , the quadratic has no real roots.

Hence the only real root of  $(f(x) = 0)$  is  $(x = -1)$ . The cubic therefore cuts the  $(x) - axis$  exactly once, not three times.