

8 ESSENTIAL QUESTIONS TO MASTER

PARTIAL FRACTIONS

1	Express $\frac{-3x^2 + 12x - 1}{(3x + 1)(x - 1)^2}$ in partial fractions.	
2	Express $\frac{2x^3 + x^2 + x}{(x - 1)(x^2 + 1)}$ in partial fractions.	
3	Express $\frac{x+7}{(x^2-9)(x-3)}$ as the sum of three partial fractions.	
4	Express $\frac{4x^2+5x-32}{(x+2)(x^2-9x-22)}$ in partial fractions.	
5	Express $\frac{5x^2+11x}{(4+x)(1+x)^2}$ in partial fractions.	
6	Express $\frac{x^2 + 4x - 2}{x^2 - x}$ in partial fractions.	
7	Divide $(x^2 - x + 1)$ by $(x^2 - 5x - 6)$. Hence express $\frac{x^2 - x + 1}{x^2 - 5x - 6}$ in partial fractions.	
8	The function $(f(x) = x^3 + ax + b)$, where a and b are constants, is exactly divisible by $(x + 3)$. Given that $(f(x))$ leaves a remainder of 56 when divided by $(x - 4)$, (i) Find the values of a and b. (ii) Determine, showing all necessary working, the number of real roots of the equation $(f(x) = 0)$.	

Answer Key:

1	We seek constants (A, B, C) so that
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$$\frac{-3x^2 + 12x - 1}{(3x + 1)(x - 1)^2}$$

$$= \frac{A}{3x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators,

$$-3x^2 + 12x - 1 = A(x - 1)^2 + B(3x + 1)(x - 1) + C(3x + 1).$$

Expand each term:

1. $A(x - 1)^2 = A(x^2 - 2x + 1) = Ax^2 - 2Ax + A$
2. $B(3x + 1)(x - 1) = B(3x^2 - 2x - 1) = 3Bx^2 - 2Bx - B$
3. $C(3x + 1) = 3Cx + C$

Summing,

$$-3x^2 + 12x - 1 = (A + 3B)x^2 + (-2A - 2B + 3C)x + (A - B + C).$$

Equating coefficients gives the system

$$A + 3B = -3$$

$$-2A - 2B + 3C = 12$$

$$A - B + C = -1$$

Solve step by step:

- From $(A + 3B = -3) \Rightarrow (A = -3 - 3B)$
- From $(A - B + C = -1)$:

$$\begin{aligned} (-3 - 3B) - B + C &= -1 \\ \Rightarrow -4B + C &= 2 \\ \Rightarrow C &= 2 + 4B. \end{aligned}$$

- Substitute A and C into $-2A - 2B + 3C = 12$:

$$\begin{aligned} -2(-3 - 3B) - 2B + 3(2 + 4B) &= 12 \\ \Rightarrow 6 + 6B - 2B + 6 + 12B &= 12 \\ \Rightarrow 16B + 12 &= 12 \\ \Rightarrow 16B &= 0 \\ \Rightarrow B &= 0. \end{aligned}$$

Then $(C = 2 + 4 \cdot 0 = 2)$ and $(A = -3 - 3 \cdot 0 = -3)$.

	Hence the partial-fraction decomposition is $\frac{-3x^2 + 12x - 1}{(3x + 1)(x - 1)^2}$ $= \frac{-3}{3x+1} + \frac{0}{x-1} + \frac{2}{(x-1)^2}$ $= -\frac{3}{3x+1} + \frac{2}{(x-1)^2}.$
2	Solution: 1. Observe that the degree of the numerator (3) is equal to the degree of the denominator (3), so first do polynomial long-division: Denominator: $((x - 1)(x^2 + 1) = x^3 - x^2 + x - 1.)$ Divide $(2x^3 + x^2 + x)$ by $(x^3 - x^2 + x - 1)$. Quotient = 2, Remainder = $((2x^3 + x^2 + x) - 2(x^3 - x^2 + x - 1))$ $(= 2x^3 + x^2 + x - (2x^3 - 2x^2 + 2x - 2))$ $(= (1 + 2)x^2 + (1 - 2)x + 2 = 3x^2 - x + 2.)$ Hence $\frac{2x^3 + x^2 + x}{(x - 1)(x^2 + 1)}$ $= 2 + \frac{3x^2 - x + 2}{(x - 1)(x^2 + 1)}.$ 2. Now decompose the proper fraction $\frac{3x^2 - x + 2}{(x - 1)(x^2 + 1)}$ $= \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$ Multiply through by $((x - 1)(x^2 + 1))$: $3x^2 - x + 2 = A(x^2 + 1) + (Bx + C)(x - 1).$ Expand the right side: $Ax^2 + A + Bx^2 - Bx + Cx - C$ $= (A + B)x^2 + (C - B)x + (A - C).$ Equate coefficients with $(3x^2 - x + 2)$:

	$A + B = 3$ $C - B = -1$ $A - C = 2.$ Solve: $A = 2, \quad B = 1, \quad C = 0.$ 3. Thus $\frac{3x^2 - x + 2}{(x-1)(x^2+1)}$ $= \frac{2}{x-1} + \frac{1 \times x + 0}{x^2+1}$ $= \frac{2}{x-1} + \frac{x}{x^2+1}$ 4. Putting it all together, $\frac{2x^3 + x^2 + x}{(x-1)(x^2+1)}$ $= 2 + \frac{2}{x-1} + \frac{x}{x^2+1}$
3	Solution. First note $x^2 - 9 = (x-3)(x+3) \Rightarrow$ $(x^2 - 9)(x-3) = (x-3)^2(x+3).$ We seek constants A,B,C so that $\frac{x+7}{(x-3)^2(x+3)}$ $= \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3}.$ Clear denominators by multiplying both sides by $((x-3)^2(x+3))$: $x+7 = A(x-3)(x+3) + B(x+3) + C(x-3)^2.$ Expand and collect like terms: $A(x^2 - 9) + B(x+3) + C(x^2 - 6x + 9)$

	$= (A + C)x^2 + (B - 6C)x + (-9A + 3B + 9C)$ $= 0 \times x^2 + 1 \times x + 7$ Equating coefficients gives the system $\begin{aligned} A + C &= 0 \\ B - 6C &= 1 \\ -9A + 3B + 9C &= 7 \end{aligned}$ From $A = -C$, the last two become $\begin{aligned} B - 6C &= 1 \\ 3B + 18C &= 7 \end{aligned}$ so $(12C = \frac{4}{3} \rightarrow C = \frac{1}{9})$, then $(B = 1 + 6C = 1 + \frac{6}{9} = \frac{5}{3})$, and $A = -\frac{1}{9}$. Hence the partial - fraction decomposition is $\begin{aligned} &\frac{x + 7}{(x^2 - 9)(x - 3)} \\ &= -\frac{\frac{1}{9}}{x - 3} + \frac{\frac{5}{3}}{(x - 3)^2} + \frac{\frac{1}{9}}{x + 3} \end{aligned}$
4	Step 1. Factor the quadratic in the denominator: $x^2 - 9x - 22 = (x - 11)(x + 2)$ so $(x + 2)(x^2 - 9x - 22) = (x + 2)^2(x - 11).$ We seek constants A, B, C such that $\begin{aligned} &\frac{4x^2 + 5x - 32}{(x + 2)^2(x - 11)} \\ &= \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{x - 11}. \end{aligned}$ Step 2. Clear denominators: $\begin{aligned} 4x^2 + 5x - 32 &= A(x + 2)(x - 11) + B(x - 11) + C(x + 2)^2. \end{aligned}$ Expand and collect like terms: $\begin{aligned} &A(x^2 - 9x - 22) + B(x - 11) + C(x^2 + 4x + 4) \\ &= (A + C)x^2 + (-9A + 4C + B)x + (-22A - 11B + 4C). \end{aligned}$

	Equate coefficients with $(4x^2 + 5x - 32)$: (1) x^2-term: $(A + C = 4)$ (2) x-term: $(-9A + 4C + B = 5)$ (3) constant: $(-22A - 11B + 4C = -32)$ Step 3. Solve the linear system: From (1) $(C = 4 - A)$. Plug into (2): $\begin{aligned} -9A + 4(4 - A) + B &= 5 \Rightarrow -9A + 16 - 4A + B = 5 \\ &\Rightarrow -13A + B = -11 \\ &\Rightarrow B = 13A - 11. \end{aligned}$ Plug B and C into (3): $\begin{aligned} -22A - 11(13A - 11) + 4(4 - A) &= -32 \\ \Rightarrow -22A - 143A + 121 + 16 - 4A &= -32 \\ \Rightarrow -169A + 137 &= -32 \\ \Rightarrow A &= 1. \end{aligned}$ Hence $(C = 4 - 1 = 3)$ and $(B = 13 \times 1 - 11 = 2)$. Step 4. Write the result: $\frac{4x^2 + 5x - 32}{(x + 2)^2(x - 11)} = \frac{1}{x + 2} + \frac{2}{(x + 2)^2} + \frac{3}{x - 11}$
5	We write $\frac{5x^2 + 11x}{(x + 4)(x + 1)^2} = \frac{A}{x + 4} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$ Clear denominators: $5x^2 + 11x = A(x + 1)^2 + B(x + 4)(x + 1) + C(x + 4)$ Expand each term: $A(x^2 + 2x + 1) + B(x^2 + 5x + 4) + Cx + 4C$ Collect like terms: $\begin{aligned} x^2: & A + B, \\ x: & 2A + 5B + C \end{aligned}$

	constant: $A + 4B + 4C$ must match $(5x^2 + 11x + 0)$. Hence the system $\begin{aligned} A + B &= 5 \\ 2A + 5B + C &= 11 \\ A + 4B + 4C &= 0 \end{aligned}$ Solve: - From $(A + B = 5) \Rightarrow (A = 5 - B)$. - Plug into $(2A + 5B + C = 11)$: $2(5 - B) + 5B + C = 11 \Rightarrow 10 - 2B + 5B + C = 11 \Rightarrow 3B + C = 1 \Rightarrow C = 1 - 3B.$ - Plug $(A = 5 - B)$ and $(C = 1 - 3B)$ into $(A + 4B + 4C = 0)$: $\begin{aligned} (5 - B) + 4B + 4(1 - 3B) &= 0 \\ \Rightarrow 5 - B + 4B + 4 - 12B &= 0 \\ \Rightarrow 9 - 9B &= 0 \\ \Rightarrow B &= 1. \end{aligned}$ Then $(A = 5 - 1 = 4)$ and $(C = 1 - 3 \times 1 = -2)$. Therefore, the partial-fraction decomposition is $\frac{5x^2 + 11x}{(x + 4)(x + 1)^2} = \frac{4}{x + 4} + \frac{1}{x + 1} + \frac{2}{(x + 1)^2}$
6	Solution: - Factor the denominator: $x^2 - x = x(x - 1).$ - Since the numerator and denominator have the same degree, do polynomial (long) division first: $\begin{aligned} &\frac{x^2 + 4x - 2}{x^2 - x} \\ &= 1 + \frac{(x^2 + 4x - 2) - (x^2 - x)}{x^2 - x} \\ &= 1 + \frac{5x - 2}{x(x - 1)}. \end{aligned}$

	3. Now set up the partial-fraction decomposition for the proper fraction: $\frac{5x - 2}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}.$ Combining the right-hand side over the common denominator gives $\begin{aligned} \frac{A}{x} + \frac{B}{x - 1} &= \frac{A(x - 1) + Bx}{x(x - 1)} \\ &= \frac{(A + B)x - A}{x(x - 1)}. \end{aligned}$ Hence we require $(A + B)x - A = 5x - 2.$ Matching coefficients: $\begin{aligned} A + B &= 5 \\ \Rightarrow \\ -A &= -2 \\ \Rightarrow \\ A &= 2, B = 3. \end{aligned}$ 4. Substitute back: $\frac{5x - 2}{x(x - 1)} = \frac{2}{x} + \frac{3}{x - 1},$ so the full decomposition is $\begin{aligned} \frac{x^2 + 4x - 2}{x^2 - x} &= 1 + \frac{2}{x} + \frac{3}{x - 1}. \end{aligned}$ Final answer $\frac{x^2 + 4x - 2}{x^2 - x} = 1 + \frac{2}{x} + \frac{3}{x - 1}$
7	1. Polynomial division. Divide $(x^2 - x + 1)$ by $(x^2 - 5x - 6)$. Since the leading terms are the same degree, the quotient is 1 and the remainder is

$$\begin{aligned}(x^2 - x + 1) - 1 \times (x^2 - 5x - 6) \\ = x^2 - x + 1 - x^2 + 5x + 6 \\ = 4x + 7\end{aligned}$$

Hence

$$\begin{aligned}\frac{x^2 - x + 1}{x^2 - 5x - 6} \\ = \\ 1 + \frac{4x + 7}{x^2 - 5x - 6}\end{aligned}$$

2. Factor the denominator.

$$x^2 - 5x - 6 = (x - 6)(x + 1)$$

So

$$\begin{aligned}\frac{x^2 - x + 1}{x^2 - 5x - 6} \\ = 1 + \frac{4x + 7}{(x - 6)(x + 1)}\end{aligned}$$

3. Partial-fraction decomposition.

We write

$$\begin{aligned}\frac{4x+7}{(x-6)(x+1)} \\ = \frac{A}{x-6} + \frac{B}{x+1}.\end{aligned}$$

Clearing denominators gives

$$\begin{aligned}4x + 7 &= A(x + 1) + B(x - 6) \\ &= (A + B)x + (A - 6B).\end{aligned}$$

Equating coefficients:

$$A + B = 4, \quad A - 6B = 7.$$

Solve:

From $A = 4 - B$, substitute into $A - 6B = 7$:

$$(4 - B) - 6B = 7$$

$$\Rightarrow$$

$$4 - 7B = 7$$

$$\Rightarrow$$

$$-7B = 3$$

$$\Rightarrow$$

$$B = -\frac{3}{7},$$

$$A = 4 - \left(-\frac{3}{7}\right) = \frac{31}{7}.$$

Therefore

$$\begin{array}{r} 4x + 7 \\ (x - 6)(x + 1) \\ \hline \frac{31}{7} \quad \frac{3}{7} \\ = \frac{31}{x - 6} - \frac{3}{x + 1}. \end{array}$$

4. Final answer.

$$\begin{array}{r} x^2 - x + 1 \\ x^2 - 5x - 6 \\ \hline = 1 \\ \frac{31}{7} \\ + \frac{3}{x - 6} \\ \hline - \frac{3}{x + 1} \end{array}$$

8 $(f(x) = x^3 + ax + b)$ is exactly divisible by $((x + 3))$, and leaves remainder 56 when divided by $((x - 4))$.

(i) Find a, b.

By the Factor Theorem, $(x + 3)$ is a factor $\Rightarrow (f(-3) = 0)$:

$$(-3)^3 + a(-3) + b = 0$$

$$\Rightarrow$$

$$-27 - 3a + b = 0$$

$$\Rightarrow$$

$$b = 27 + 3a.$$

Also, the Remainder Theorem says ($f(4) = 56$):

$$\begin{aligned}
 4^3 + a \cdot 4 + b &= 56 \\
 \Rightarrow \\
 64 + 4a + b &= 56
 \end{aligned}$$

Substitute ($b = 27 + 3a$):

$$\begin{aligned}
 64 + 4a + (27 + 3a) &= 56 \\
 \Rightarrow \\
 91 + 7a &= 56 \\
 \Rightarrow \\
 7a &= -35 \\
 \Rightarrow \\
 a &= -5.
 \end{aligned}$$

Then

$$b = 27 + 3(-5) = 27 - 15 = 12.$$

Answer (i): ($a = -5$, $b = 12$)

(ii) Number of real roots of ($f(x) = 0$).

Now

$$f(x) = x^3 - 5x + 12.$$

Compute its discriminant for a cubic ($x^3 + px + q$):

$$\begin{aligned}
 \Delta &= -4p^3 - 27q^2, \\
 p &= -5, q = 12 \\
 \Rightarrow \\
 \Delta &= -4(-5)^3 - 27(12)^2 \\
 &= -4(-125) - 27 \cdot 144 \\
 &= 500 - 3888 \\
 &= -3388 < 0.
 \end{aligned}$$

A cubic with negative discriminant has exactly one real root (and two nonreal complex conjugates).

Answer (ii): There is exactly one real root.