

8 ESSENTIAL QUESTIONS TO MASTER

NATURE OF ROOTS

1	Let the graph of the curve be $y = x^2 + (3k - 2)x + (5k - 1)$, Where k is a real constant. (a) For which values of k does the curve lie entirely above the x-axis (i.e. $y > 0$) for all real x? (b) For which values of k does the line $y = x + 3$ intersect the curve in (real) points?	
2	(a) Solve the inequality $4^x + 2 > 3x^2$. (b) Prove that for every real parameter (m), the straight line $y = 2x + 1$ meets the parabola $y = mx^2 + 4x - m$ in two distinct real points.	
3	Determine the nature of the roots of the following equations. (a) $(x^2 + 3x - 5 = 0)$ (b) $(1 - 4x + 4x^2 = 0)$ (c) $(2 + 5x^2 = 3x)$	
4	It is given that the quadratic equation $(p + 5)x^2 - 12x + (2p + 4) = 0$ has equal real roots. Find the possible values of (p).	
5	(a) Find the range of values of p for which the equation $(x^2 + (p - 2)x = p - 10)$ has real roots. (b) Find the range of values of q for which the equation $(x^2 + 1 = (4 - q)x)$ has no real roots.	
6	Find the range of values of k for which the equation $x^2 + 2k + 10 = x - 3kx$ is always positive for all real values of x.	
7	Given that x is real, show that $(\frac{x^2+2}{2x+1})$ cannot lie between -2 and 1.	
8	Does the equation $x^3 + 8 = 0$? Justify your answer.	

Answer Key:

- 1 (a) Curve entirely above the x-axis

A quadratic ($y = x^2 + Bx + C$) lies above the x-axis for all x exactly when:

1. Its leading coefficient is positive ($1 > 0$, already true).
2. Its discriminant is negative:

$$\Delta = B^2 - 4 \cdot 1 \cdot C < 0.$$

Here $B=3k-2$ and $C=5k-1$. So

$$\begin{aligned}\Delta &= (3k - 2)^2 - 4(5k - 1) \\ &= 9k^2 - 12k + 4 - 20k + 4 \\ &= 9k^2 - 32k + 8.\end{aligned}$$

We require $9k^2 - 32k + 8 < 0$.

Solve the quadratic inequality:

1. Find roots of $(9k^2 - 32k + 8 = 0)$.

$$\begin{aligned}k &= \frac{32 \pm \sqrt{(-32)^2 - 4 \times 9 \times 8}}{2 \cdot 9} \\ &= \frac{32 \pm \sqrt{1024 - 288}}{18} \\ &= \frac{32 \pm \sqrt{736}}{18} \\ &= \frac{32 \pm 4\sqrt{46}}{18} \\ &= \frac{16 \pm 2\sqrt{46}}{9}.\end{aligned}$$

Numerically,

$$k_1 = \frac{16 - 2\sqrt{46}}{9},$$

$$k_2 = \frac{16 + 2\sqrt{46}}{9},$$

with $(k_1 < k_2)$.

2. Since the leading coefficient 9 is positive, $(9k^2 - 32k + 8 < 0)$ for $k_1 < k < k_2$.

Hence

$$\frac{16 - 2\sqrt{46}}{9} < k < \frac{16 + 2\sqrt{46}}{9}$$

is the range for which the curve stays strictly above the x-axis.

(b) Intersection with ($y = x + 3$)

An intersection satisfies

$$x^2 + (3k - 2)x + (5k - 1) = x + 3.$$

Bring all terms to one side:

$$\begin{aligned} x^2 + (3k - 2 - 1)x + (5k - 1 - 3) &= 0 \\ \Rightarrow \\ x^2 + (3k - 3)x + (5k - 4) &= 0. \end{aligned}$$

This quadratic in x has real solutions exactly when its discriminant is nonnegative:

$$\Delta' = (3k - 3)^2 - 4(5k - 4) \geq 0.$$

Compute:

$$\begin{aligned} \Delta' &= 9k^2 - 18k + 9 - 20k + 16 \\ &= 9k^2 - 38k + 25. \end{aligned}$$

Require

$$9k^2 - 38k + 25 \geq 0.$$

Find its roots:

$$\begin{aligned} k &= \frac{38 \pm \sqrt{38^2 - 4 \times 9 \times 25}}{18} \\ &= \frac{38 \pm \sqrt{1444 - 900}}{18} \\ &= \frac{38 \pm \sqrt{544}}{18} \\ &= \frac{38 \pm 4\sqrt{34}}{18} \\ &= \frac{19 \pm 2\sqrt{34}}{9}. \end{aligned}$$

Call them (k_-) and (k_+) with

	$k_- = \frac{19 - 2\sqrt{34}}{9},$ $k_+ = \frac{19 + 2\sqrt{34}}{9}.$ Since the leading coefficient $9 > 0$, the quadratic $(9k^2 - 38k + 25)$ is nonnegative for $k \leq k_-$ or $k \geq k_+.$ Thus the line $(y = x + 3)$ meets the curve in real points exactly when $k \leq \frac{19 - 2\sqrt{34}}{9}$ or $k \geq \frac{19 + 2\sqrt{34}}{9}.$
2	(a) Find all real (x) satisfying $4^x + 2 > 3x^2$. Define $f(x) = 4^x + 2 - 3x^2.$ We look for the zeros of $(f(x))$ and test the sign. Numerically one finds two simple real roots $x_1 \approx -0.53$, $x_2 \approx 1.30$. Since (4^x) grows rapidly as $(x \rightarrow +\infty)$ and $(f(x) \rightarrow -\infty)$ as $(x \rightarrow -\infty)$, one checks that $(f(x) > 0)$ exactly for $x \in (x_1, x_2)$, i.e. $\boxed{-0.53 < x < 1.30}.$ (b) To find intersection points solve $2x + 1 = mx^2 + 4x - m$. Rearrange into standard quadratic form in (x): $mx^2 + (4 - 2)x - (m + 1) = 0$ $\Rightarrow$ $mx^2 + 2x - (m + 1) = 0.$ This is a quadratic in (x). Its discriminant is $\Delta = 2^2 - 4(m) [-(m + 1)]$ $= 4 + 4m(m + 1)$ $= 4(1 + m^2 + m)$ $= 4(m^2 + m + 1)$

	But for all real (m), $m^2 + m + 1 = \left(m + \frac{1}{2}\right)^2 + \frac{3}{4} > 0.$ Hence ($\Delta > 0$). A quadratic with positive discriminant has two distinct real roots. Therefore, for every real (m), the line ($y = 2x + 1$) meets the given parabola in exactly two distinct real points.
3	(a) ($x^2 + 3x - 5 = 0$) Discriminant ($D = b^2 - 4ac = 3^2 - 4 \times 1 \times (-5) = 9 + 20 = 29 > 0$) Hence the roots are real and different. (b) ($1 - 4x + 4x^2 = 0$) Rewrite in standard form: ($4x^2 - 4x + 1 = 0$) Discriminant ($D = (-4)^2 - 4 \times 4 \times 1 = 16 - 16 = 0$.) Hence the roots are real and equal. (c) ($2 + 5x^2 = 3x$) Rewrite: ($5x^2 - 3x + 2 = 0$) Discriminant ($D = (-3)^2 - 4 \times 5 \times 2 = 9 - 40 = -31 < 0$.) Hence the roots are not real (complex conjugates).
4	We have the quadratic equation $(p + 5)x^2 - 12x + (2p + 4) = 0.$ Compare with ($ax^2 + bx + c = 0$), so $a = p + 5, \quad b = -12, \quad c = 2p + 4.$ For equal real roots, the discriminant ($D = b^2 - 4ac$) must be zero: $(-12)^2 - 4(p + 5)(2p + 4) = 0.$ Compute: $144 - 4((p + 5)(2p + 4)) = 0$ $\Rightarrow 144 - 4(2p^2 + 10p + 4p + 20) = 0$ $144 - 4(2p^2 + 14p + 20) = 0$ $\Rightarrow 144 - (8p^2 + 56p + 80) = 0$ $144 - 8p^2 - 56p - 80 = 0$ $\Rightarrow 64 - 8p^2 - 56p = 0$ $\Rightarrow$

	$8 - p^2 - 7p = 0$ Multiply both sides by -1: $p^2 + 7p - 8 = 0.$ Factor: $(p - 1)(p + 8) = 0$ $\Rightarrow$ $p = 1 \quad \text{or} \quad p = -8.$ Thus the possible values of (p) are $(1 \text{ or } -8)$
5	Here are the two results (with brief derivations): (a) Equation $(x^2 + (p - 2)x = p - 10)$ can be written as $x^2 + (p - 2)x + [-(p - 10)] = 0,$ so its discriminant is $D = (p - 2)^2 - 4 \times 1 \times [-(p - 10)]$ $= (p - 2)^2 + 4(p - 10)$ $= p^2 - 4p + 4 + 4p - 40$ $= p^2 - 36$ For real roots we need $(D \geq 0)$, i.e. $p^2 - 36 \geq 0$ $\Rightarrow$ $p \leq -6 \quad \text{or} \quad p \geq 6.$ (b) Equation $(x^2 + 1 = (4 - q)x)$ is $x^2 - (4 - q)x + 1 = 0$ with discriminant $D = (4 - q)^2 - 4 \times 1 \times 1$ $= q^2 - 8q + 16 - 4$ $= q^2 - 8q + 12$ To have no real roots we require $D < 0$. The quadratic $(q^2 - 8q + 12)$ factors (or has roots at $(q = 2, 6)$), and being upward-opening,

	$q^2 - 8q + 12 < 0$ $\Rightarrow$ $2 < q < 6.$ Final answer: • For real roots of the first equation: $(p \leq -6)$ or $(p \geq 6)$. • For no real roots of the second: $(2 < q < 6)$.
6	Define $f(x) = (x^2 + 2k + 10) - (x - 3kx)$ $= x^2 + (3k - 1)x + (2k + 10)$ We require 1) Leading coefficient $(a = 1 > 0)$ (already true), and 2) Discriminant $(D < 0)$. Compute $D = (3k - 1)^2 - 4 \cdot 1 \cdot (2k + 10)$ $= 9k^2 - 6k + 1 - 8k - 40$ $= 9k^2 - 14k - 39$ So the condition $D < 0$ becomes $9k^2 - 14k - 39 < 0.$ Solve $(9k^2 - 14k - 39 = 0)$: $k = \frac{14 \pm \sqrt{(-14)^2 - 4 \cdot 9 \cdot (-39)}}{2 \times 9}$ $= \frac{14 \pm \sqrt{196 + 1404}}{18}$ $= \frac{14 \pm 40}{18}$ which gives the two roots $k = -\frac{13}{9} \text{ \& } k = 3.$ Because the quadratic $(9k^2 - 14k - 39)$ opens upward, it is negative exactly between its roots. Hence the required range is $-\frac{13}{9} < k < 3$.
7	Here is a succinct proof that for every real (x), the fraction $f(x) = \frac{x^2 + 2}{2x + 1}$ never falls into the open interval $((-2, 1))$.

1. Show $(f(x) > 2)$ for all real $(x \neq -2)$:

$$f(x) > 2$$

$$\Leftrightarrow$$

$$\frac{x^2 + 2}{2x + 1} + 2 > 0$$

$$\Leftrightarrow$$

$$\frac{x^2 + 2 + 2(2x + 1)}{2x + 1} > 0$$

$$\Leftrightarrow$$

$$\frac{x^2 + 4x + 4}{2x + 1} > 0$$

$$\Leftrightarrow$$

$$\frac{(x + 2)^2}{2x + 1} > 0$$

Now $((x + 2)^2 \geq 0)$ always, and the only point where $((x + 2)^2 = 0)$ is $(x = -2)$, at which $(2x + 1 = -3 \neq 0)$, so the whole fraction is zero there. Everywhere else the numerator is positive and the denominator $(2x + 1)$ changes sign at $(x = -\frac{1}{2})$, but one checks directly that the sign of numerator and denominator coincide on each interval $((-\infty, -\frac{1}{2}))$ and $((-\frac{1}{2}, +\infty))$.

Hence

$$f(x) \geq -2 \quad \{\text{with equality only at } x = -2,\}$$

and in particular $(f(x) > 2)$ for every $(x \neq -2)$.

2. Show $(f(x) < 1)$ has no real solution:

$$f(x) < 1$$

$$\Leftrightarrow$$

$$\frac{x^2 + 2}{2x + 1} < 1$$

$$\Leftrightarrow$$

$$x^2 + 2 < 2x + 1$$

$$\Leftrightarrow$$

$$(x - 1)^2 < 0,$$

which is impossible for any real (x) . The best you can do is equality $((x - 1)^2 = 0)$ $(x = 1)$, giving $(f(1) = 1)$.

Putting these two facts together:

- $(f(x) > 2)$ for all real $(x \neq -2)$, and $(f(-2) = -2)$.

- $(f(x) \geq 1)$ only at $(x = 1)$, and $(f(x) > 1)$ elsewhere (since $(f(x) < 1)$ has no solutions).

	Therefore, there is no real x for which $-2 < \frac{x^2+2}{2x+1} < 1.$ In other words, $\frac{x^2+2}{2x+1}$ cannot lie between -2 and 1.
8	1. Rewrite the equation We want $x^3 + 8 = 0$ $\Rightarrow$ $x^3 = -8.$ 2. Recognize sum of cubes Notice that $(8 = 2^3)$. The left side is a sum of cubes: $x^3 + 2^3 = 0.$ A standard factorization is $a^3 + b^3 = (a + b)(a^2 - ab + b^2).$ Here $(a = x)$ and $(b = 2)$, so $x^3 + 2^3$ $= (x + 2)(x^2 - 2x + 2^2)$ $= (x + 2)(x^2 - 2x + 4).$ 3. Solve each factor = 0 1. Factor $(x + 2 = 0)$ $x + 2 = 0$ $\Rightarrow$ $x = -2.$ 2. Factor $(x^2 - 2x + 4 = 0)$ Compute its discriminant: $D = (-2)^2 - 4 \times 1 \times 4 = 4 - 16 = -12 < 0.$ Since the discriminant is negative, this quadratic has no real solutions (it has two complex ones). 4. Conclusion - Real solution: $(x = -2)$. - Complex solutions: $(x = \frac{2 \pm i\sqrt{12}}{2} = 1 \pm i\sqrt{3})$.

Thus, in the real numbers the only solution of

$$x^3 + 8 = 0$$

is

$$x = -2.$$