

7 ESSENTIAL QUESTIONS TO MASTER

LOGARITHM

1	Given that $(\log_4 x = p)$, express each of the following in terms of p : (a) $(\log_4(4x^2))$ (b) $(\log_4 \frac{16}{x})$ (c) $(\log_2 2x^2)$ (d) $5\sqrt{x}$	
2	Solve the following pairs of simultaneous equations. (a) $\log_4(x - 2y) = 0$ $\lg(x - 1) = 2 \lg y$ (b) $\log_3(3x - y + 5) = 2$ $\ln(11 - y) - \ln x = \ln 2$	
3	Express $(2 \log x - \log(x - 8) = 2)$ as a quadratic equation in (x) and explain why there are no real solutions.	
4	(a) Without using a calculator, show that $2 + \frac{1}{\log_3 2} \frac{\ln 7}{\ln 2} = \log_3 84.$ (b) Without using a calculator, show that $\log_5 8 \times \log_{128} 3 \times \log_{81} 125 = \frac{9}{28}.$	
5	The safety index of a particular activity is defined as the logarithm of p , where the fatality rate of that activity is one in p people. A study has found that there is a one in 12 500 chance of dying from electrocution. (i) State the safety index. (ii) Given that the safety index of another activity is twice the answer in part (i), is it correct to conclude that there is a one in 6 250 chance of a fatality resulting from this activity? Explain your answer with calculations.	
6	(a) It is given that $1 - \ln y = \ln(x + y).$ Express x in terms of y .	

	(b) Solve the equation $6 \log_y 2 = 5 - \log_2 y$.	
7	Express the equation $\log_2 x + \log_4(x - 1) = 3$ as a cubic equation in x .	

Answer Key:

1	(a) $\begin{aligned} \log_4(4x^2) \\ &= \log_4 4 + \log_4 x^2 \\ &= 1 + 2 \log_{\{4\}} x \\ &= 1 + 2p \end{aligned}$ (b) $\begin{aligned} \log_4 \left(\frac{16}{x} \right) \\ &= \log_4 16 - \log_4 x \\ &= 2 - p. \end{aligned}$ (c) $\begin{aligned} \log_2(2x) \\ &= \log_2 2 + \log_2 x \\ &= 1 + \log_2 x \\ &= 1 + 2p \\ &\Rightarrow \\ (\log_2(2x))^2 &= (1 + 2p)^2. \end{aligned}$ (d) $\begin{aligned} 5\sqrt{x} \\ &= 5x^{1/2} \\ &= 5(4^p)^{1/2} \\ &= 5 \times 2^p \end{aligned}$	
2	Equations: $(\log_4(x - 2y) = 0)$ $(\lg(x - 1) = 2 \lg y)$ 1. From $(\log_4(x - 2y) = 0)$ we get $x - 2y = 4^0 = 1$. 2. From $(\lg(x - 1) = 2 \lg y)$ (where $\lg$ means $(\log_{\{10\}})$) we use the power-rule: $\lg(x - 1) = \lg(y^2)$	

$$\Rightarrow$$

$$x - 1 = y^2,$$

$$\text{So } x = y^2 + 1.$$

3.

$$(y^2 + 1) - 2y = 1$$

$$y^2 - 2y + 1 = 1$$

$$y^2 - 2y = 0$$

$$y(y - 2) = 0$$

Reject $y=0$ (because $\lg y$ would be undefined). Hence $y=2$.
 Then from $(x = y^2 + 1)$ we get $(x = 2^2 + 1 = 5)$.

Answer (a):

$$(x, y) = (5, 2).$$

(b)

Equations:

$$\log_3(3x - y + 5) = 2$$

$$\ln(11 - y) - \ln x = \ln 2$$

1. From $(\log_3(3x - y + 5) = 2)$ we get

$$3x - y + 5 = 3^2 = 9$$

$\Rightarrow$

$$3x - y = 4.$$

2. From $\ln(11 - y) - \ln x = \ln 2$

we combine logs:

$$\ln \left(\frac{|11 - y|}{x} \right) = \ln 2$$

$\Rightarrow$

$$\frac{|11 - y|}{x} = 2$$

$\Rightarrow$

$$|11 - y| = 2x.$$

We now split (2') into two cases:

Case I: $(11 - y = 2x)$.

Then $(y = 11 - 2x)$. Substitute into (1'):

$$3x - (11 - 2x) = 4$$

$\Rightarrow$

$$3x - 11 + 2x = 4$$

$\Rightarrow$

	$5x=15$ $\Rightarrow$ $x = 3,$ $y = 11 - 2 \times 3 = 5.$ Check: $\frac{11-5}{3} = \frac{6}{3} = 2 \rightarrow \text{OK}.$ Case II: $(y - 11 = 2x).$ Then $(y = 11 + 2x).$ $3x - (11 + 2x) = 4$ $\Rightarrow$ $3x - 11 - 2x = 4$ $\Rightarrow$ $x = 15,$ $y = 11 + 2 \times 15 = 41.$ Check: $\frac{11-41}{15} = \frac{30}{15} = 2 \rightarrow \text{OK}.$ Answer (b): $(x, y) = (3, 5) \text{ or } (15, 41).$
3	1. Start with $2 \log_2 x - \log_2(x - 8) = 2.$ 2. Use the power-rule and quotient-rule of logarithms: $2 \log_2 x = \log_2(x^2),$ so $\log_2(x^2) - \log_2(x - 8) = \log_2(x^2/(x - 8)).$ Hence the equation becomes $\log_2(x^2/(x - 8)) = 2.$ 3. Rewrite in exponential form: $x^2/(x - 8) = 2^2 = 4.$ 4. Multiply both sides by $(x - 8):$ $x^2 = 4(x - 8) = 4x - 32$ $\Rightarrow x^2 - 4x + 32 = 0.$ 5. Compute the discriminant Δ of the quadratic $x^2 - 4x + 32 = 0:$

	$\Delta = (-4)^2 - 4 \cdot 1 \cdot 32 = 16 - 128 = -112.$ Since $\Delta < 0$, there are no real solutions. Answer: The equation is equivalent to $x^2 - 4x + 32 = 0,$ and because its discriminant is negative, it has no real roots.
4	Below are full without-calculator" verifications of each identity. 1. (Showing the first identity) The printed statement $2 + \frac{1}{\log_3 2} \frac{\ln 7}{\ln 2}$ $= \log_3 84$ as written is numerically false (you can check each side approximately). I believe there is a misprint. A correct and typical exercise is: Show without a calculator that $2 + \frac{\ln 7}{\ln 3} = \log_3(63).$ Here is the proof of that corrected version: 1. Notice ($2 = \log_3 9$). 2. Also $\frac{\ln 7}{\ln 3} = \log_3 7$ 3. Hence $2 + \frac{\ln 7}{\ln 3} = \log_3 9 + \log_3 7$ $= \log_3(9 \times 7)$ $= \log_3(63)$ 2. (Showing $\log_5 8 \times \log_{128} 3 \times \log_{81} 125 = \frac{9}{28}$) Use the change-of-base formula in terms of natural logarithms: $\log_5 8 = \frac{\ln 8}{\ln 5},$

	$\frac{\log_{128} 3}{\ln 3}$ $= \frac{\ln 128}{\ln 3},$ $\log_{81} 125 = \frac{\ln 125}{\ln 81}.$ Therefore the product is $\log_5 8 \log_{128} 3 \log_{81} 125$ $= \frac{\ln 8}{\ln 5} \frac{\ln 3}{\ln 128} \frac{\ln 125}{\ln 81}$ $= \frac{\ln 8 \ln 3 \ln 125}{\ln 5 \ln 128 \ln 81}.$ Now substitute the prime-power factorizations: $-(8 = 2^3 \rightarrow \ln 8 = 3 \ln 2).$ $-(128 = 2^7 \rightarrow \ln 128 = 7 \ln 2).$ $-(125 = 5^3 \rightarrow \ln 125 = 3 \ln 5).$ $-(81 = 3^4 \rightarrow \ln 81 = 4 \ln 3).$ Hence $\ln 8 \ln 3 \ln 125$ $= (3 \ln 2)(\ln 3)(3 \ln 5)$ $= 9 \ln 2 \ln 3 \ln 5,$ and $\ln 5 \ln 128 \ln 81$ $= (\ln 5)(7 \ln 2)(4 \ln 3)$ $= 28 \ln 2 \ln 3 \ln 5.$ Dividing, $\frac{9 \ln 2 \ln 3 \ln 5}{28 \ln 2 \ln 3 \ln 5}$ $= \frac{9}{28}.$ Thus indeed $\log_5 8 \times \log_{128} 3 \times \log_{81} 125 = \frac{9}{28}.$
5	(i) By definition the safety index is $S = \log_{10}(p),$ where “one in p people” die. Here p=12 500, so

	$ \begin{aligned} S &= \log_{10}(12\,500) \\ &= \log_{10}(1.25 \times 10^4) \\ &= \log_{10}(1.25) + 4 \\ &\approx 0.09691 + 4 = 4.09691 \approx 4.10. \end{aligned} $ (ii) Suppose another activity has safety index $2S$. Then its fatality parameter p satisfies $ \begin{aligned} 2S &= \log_{10}(p') \\ &\Rightarrow \\ p' &= 10^{2S} = (10^S)^2 \\ &= (12\,500)^2 \\ &= 156\,250\,000. \end{aligned} $ Thus the chance of dying is one in $(156\,250\,000)$, not one in 6250. In other words, doubling the logarithm doubles the exponent on 10, so you square the original p. The conclusion “one in $6\,250$” is incorrect.
6	(a) Solve $1 - \ln y = \ln(x + y)$. Step 1. Exponentiate both sides: $e^{1 - \ln y} = x + y.$ Step 2. Rewrite the left side: $ \begin{aligned} e^1 e^{-\ln y} &= e \times \frac{1}{y} \Rightarrow \\ \frac{e}{y} &= x + y. \end{aligned} $ Step 3. Rearrange for (x): $x = \frac{e}{y} - y.$ Answer (a): $x = \frac{e}{y} - y$ (b) Solve $6 \log_y 2 = 5 - \log_2 y$. Step 1. Let $(t = \log_2 y)$. Then $(\log_y 2 = \frac{1}{t}.)$ The equation becomes $6 \frac{1}{t} = 5 - t.$

	Step 2. Multiply both sides by t: $6 = 5t - t^2$ $\Rightarrow$ $t^2 - 5t + 6 = 0.$ Step 3. Factor and solve: $(t - 2)(t - 3) = 0$ $\Rightarrow t = 2 \quad \text{or} \quad t = 3.$ Step 4. Recall ($t = \log_2 y$): $- \text{If } (t = 2), \text{ then } (y = 2^2 = 4.)$ $- \text{If } (t = 3), \text{ then } (y = 2^3 = 8.)$ Answer (b): ($y = 4$) or ($y = 8$.)
7	Starting from $\log_2 x + \log_4(x - 1) = 3,$ note that $(\log_4(x - 1) = \frac{\log_2(x-1)}{\log_2(4)} = \frac{1}{2}\log_2(x - 1))$. Hence 1. Rewrite the equation: $\log_2 x + \frac{1}{2}\log_2(x - 1) = 3.$ 2. Multiply through by 2: $2\log_2 x + \log_2(x - 1) = 6.$ 3. Combine the logarithms: $\log_2(x^2(x - 1)) = 6.$ 4. Exponentiate base 2: $x^2(x - 1) = 2^6 = 64.$ 5. Expand to obtain the cubic: $x^3 - x^2 - 64 = 0.$ So the equation in standard cubic form is $\boxed{x^3 - x^2 - 64 = 0},$ with the understanding that $x > 1$ so that the logarithms are defined.