

8 ESSENTIAL QUESTIONS TO MASTER

EXPONENTIAL

1	Given that $\frac{16^y}{9^y} = \frac{2^{4y}}{3^{2y}} = \frac{4^{3-y}}{27^y}$, find the value of (12^y) .	
2	Solve (i) $(2(4^x) - 4^{2-x} = 6)$ (ii) $(5^x = e^{2x+3})$	
3	It is given that $(3^{4-y} \times 5^{y+1} = 5^{2y+3})$. (i) Show that $15^y = \frac{81}{25^2}$ (ii) Hence, find the value of y, correct to 2 decimal places.	
4	Given that $(\sqrt{125^x} = \frac{5^{1-x}}{25^x})$, find the value of $(\sqrt{125^x})$.	
5	Without using a calculator, find the value of x such that $9(3^x)^3 = \sqrt[3]{\frac{1}{27^x}}$.	
6	(a) The percentage, P, of carbon-14 remaining in a piece of fossilised wood is given by $(P = 100e^{-kt})$, where k is a constant and t is measured in years. It takes 5730 years for the carbon-14 to be reduced to half of the original amount. Calculate (i) the value of k (ii) the percentage of carbon-14 which would indicate a fossil age of 8000 years. (b) The size, S, and intensity, I, of a naturally occurring event are connected by the formula $(S = \lg \frac{I}{c})$, where c is a constant. Calculate, to 1 decimal place, the size of the event which has intensity 50 times that of an event of size 2.4.	
7	(i) Given that $(y = Ae^{px})$, where A and p are constants, find an expression for $(\frac{dy}{dx})$. (ii) Hence, find the value of A and of p for which $(\frac{dy}{dx} + 4y = 14e^{3x})$.	
8	The population of a species of insects in a colony, N insects, after t days, can be modelled by $(N = 10\,000e^{-0.2t})$. (i) Find the initial population of the insects. (ii) Estimate the number of insects after 5 days. (iii) When will the colony first decrease to 1000 insects?	

Answer Key:

1	Given:	
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	$\frac{16^y}{9^y} = \frac{4^{3-y}}{27^y}$ We are to find the value of (12^y). Step 1: Express all terms with base 2 and 3: • $(16 = 2^4)$ • $(9 = 3^2)$ • $(4 = 2^2)$ • $(27 = 3^3)$ So, $\frac{16^y}{9^y} = \frac{(2^4)^y}{(3^2)^y} = \frac{2^{4y}}{3^{2y}}$ $\frac{4^{3-y}}{27^y} = \frac{(2^2)^{3-y}}{(3^3)^y} = \frac{2^{2(3-y)}}{3^{3y}} = \frac{2^{6-2y}}{3^{3y}}$ Step 2: Set the expressions equal: $\frac{2^{4y}}{3^{2y}} = \frac{2^{6-2y}}{3^{3y}}$ Step 3: Equate numerators and denominators: $2^{4y} = 2^{6-2y} \rightarrow 4y = 6 - 2y \rightarrow 6y = 6 \rightarrow y = 1$ $3^{2y} = 3^{3y} \rightarrow 2y = 3y \rightarrow y = 0$ But from the first equation, $y = 1$. Step 4: Find (12^y): $12^y = (2^2 \times 3)^y = 2^{2y} \times 3^y$ When $y = 1$: $12^1 = 2^{2 \times 1} \times 3^1 = 2^2 \times 3 = 4 \times 3 = 12$	
2	(i) 1. Rewrite (4^{2-x}) as $(4^2 \cdot 4^{-x} = 16 \cdot 4^{-x})$. 2. Let $(y = 4^x)$, so $(4^{-x} = \frac{1}{y})$. Substitute into the equation: $2y - 16 \cdot \frac{1}{y} = 6$ Multiply both sides by y: $2y^2 - 16 = 6y$ $2y^2 - 6y - 16 = 0$	

	Divide by 2: $y^2 - 3y - 8 = 0$ Factor: $(y - 4)(y + 2) = 0$ So, $y = 4$ or $y = -2$. But ($y = 4^x > 0$), so ($y = 4$). Thus, $4^x = 4 \rightarrow x = 1$ (ii) Take the natural logarithm of both sides: $\ln(5^x) = \ln(e^{2x+3})$ $x \ln 5 = 2x + 3$ $x \ln 5 - 2x = 3$ $x(\ln 5 - 2) = 3$ $x = \frac{3}{\ln 5 - 2}$	
3	(i) Show that ($15^y = \frac{81}{25^2}$): Start with the given equation: $3^{4-y} \times 5^{y+1} = 5^{2y+3}$ Rewrite (5^{2y+3}) as ($5^{y+1} \times 5^{y+2}$): $3^{4-y} \times 5^{y+1} = 5^{y+1} \times 5^{y+2}$ Divide both sides by (5^{y+1}): $3^{4-y} = 5^{y+2}$ Take logarithms or rewrite as: $3^{4-y} = 5^{y+2}$ Express both sides with base 15: $3^{4-y} = (3 \times 5)^y$ But this is not directly possible, so let's manipulate further. Alternatively, bring all terms to one side: $3^{4-y} \times 5^{-(y+2)} = 1$ $\frac{3^{4-y}}{5^{y+2}} = 1$ But to match the form (15^y), let's try multiplying both sides by (3^y) and (5^{-y}):	

	$3^{4-y} \times 5^{y+1} = 5^{2y+3}$ $3^{4-y} = 5^{2y+3} \times 5^{-(y+1)}$ $3^{4-y} = 5^{y+2}$ Now, multiply both sides by (3^y): $3^{4-y} \times 3^y = 3^4 = 5^{y+2} \times 3^y$ $3^4 = 3^y \times 5^{y+2}$ $3^4 = (3 \times 5)^y \times 5^2$ $3^4 = 15^y \times 25$ $\frac{3^4}{25} = 15^y$ $\frac{81}{25} = 15^y$ (ii) Find the value of y, correct to 2 decimal places: $15^y = \frac{81}{25}$ Take logarithms on both sides: $y \log 15 = \log 81 - \log 25$ $y = \frac{\log 81 - \log 25}{\log 15}$ Now, calculate: $-(\log 81 = \log(3^4) = 4 \log 3 \approx 4 \times 0.4771 = 1.9084)$ $-(\log 25 = \log(5^2) = 2 \log 5 \approx 2 \times 0.6990 = 1.3980)$ $-(\log 15 = \log(3 \times 5) = \log 3 + \log 5 \approx 0.4771 + 0.6990 = 1.1761)$ So, $y = \frac{1.9084 - 1.3980}{1.1761} = \frac{0.5104}{1.1761} \approx 0.43$	
4	Step-by-Step Solution: Let's start by simplifying both sides. 1. Simplify the left side $\sqrt{125^x} = (125^x)^{1/2} = 125^{x/2}$ Since $(125 = 5^3)$: $125^{x/2} = (5^3)^{x/2} = 5^{3x/2}$	

	2. Simplify the right side $\frac{5^{1-x}}{25^x}$ Since $(25 = 5^2)$: $25^x = (5^2)^x = 5^{2x}$ So, $\frac{5^{1-x}}{5^{2x}} = 5^{1-x-2x} = 5^{1-3x}$ 3. Set the simplified expressions equal $5^{3x/2} = 5^{1-3x}$ Since the bases are the same, set the exponents equal: $\frac{3x}{2} = 1 - 3x$ Multiply both sides by 2 to clear the fraction: $3x = 2 - 6x$ Add $6x$ to both sides: $3x + 6x = 2$ $9x = 2$ $x = \frac{2}{9}$ 4. Find the value of $(\sqrt{125^x})$ Recall from earlier: $\sqrt{125^x} = 5^{\frac{3x}{2}}$ Substitute $(x = \frac{2}{9})$: $5^{3 \times \frac{2}{9} / 2} = 5^{\frac{6 \times 1}{18}} = 5^{\frac{6}{18}} = 5^{\frac{1}{3}}$ So, $\sqrt{125^x} = 5^{\frac{1}{3}}$	
5	Given: $9(3^x)^3 = \sqrt[3]{\frac{1}{27^x}}$ Step 1: Simplify the left side $(3^x)^3 = 3^{3x}$ So, $9(3^x)^3 = 9 \times 3^{3x}$	

	But ($9 = 3^2$), so: $9 \times 3^{3x} = 3^2 \times 3^{3x} = 3^{2+3x}$ Step 2: Simplify the right side $27^x = (3^3)^x = 3^{3x}$ So, $\frac{1}{27^x} = 3^{-3x}$ Now, $\sqrt[3]{3^{-3x}} = (3^{-3x})^{1/3} = 3^{-x}$ Step 3: Set the simplified sides equal $3^{2+3x} = 3^{-x}$ Since the bases are the same, set the exponents equal: $2 + 3x = -x$ Step 4: Solve for (x) $2 + 3x = -x$ $2 = -x - 3x$ $2 = -4x$ $x = -\frac{1}{2}$	
6	(a) Carbon-14 Decay The formula is: $P = 100e^{-kt}$ (i) Find the value of k Given: - After 5730 years, half the original amount remains, so $P = 50$ when $t = 5730$. Substitute into the formula: $50 = 100e^{-k \times 5730}$ Divide both sides by 100: $0.5 = e^{-5730k}$	

Take the natural logarithm of both sides:

$$\ln(0.5) = -5730k$$

$$-0.6931 = -5730k$$

$$k = \frac{0.6931}{5730}$$

$$k \approx 0.000121$$

(ii) Percentage of carbon-14 after 8000 years

Use the formula with ($t = 8000$) and ($k \approx 0.000121$):

$$P = 100e^{-0.000121 \times 8000}$$

Calculate the exponent:

$$-0.000121 \times 8000 = -0.968$$

$$P = 100e^{-0.968}$$

$$P \approx 100 \times 0.379$$

$$P \approx 37.9\%$$

(b) Size and Intensity of an Event

$$\text{Given } S = \lg \frac{I}{c}$$

Let the first event have size ($S_1 = 2.4$) and intensity (I_1).

The second event has intensity ($I_2 = 50I_1$), and we are to find its size (S_2).

First, express (I_1) in terms of (c) and (S_1):

$$S_1 = \lg \frac{I_1}{c}$$

$$= \frac{I_1}{c}$$

$$I_1 = c \times 10^{S_1}$$

$$\text{Now, } (I_2 = 50I_1 = 50c \times 10^{S_1}).$$

So,

$$S_2 = \lg \frac{I_2}{c} = \lg \frac{50c \times 10^{S_1}}{c} = \lg(50 \times 10^{S_1})$$

$$= \lg 50 + \lg 10^{S_1}$$

$$= \lg 50 + S_1$$

Now, ($\lg 50 \approx 1.699$), and ($S_1 = 2.4$):

	$S_2 = 1.699 + 2.4 = 4.099$ Rounded to 1 decimal place: $S_2 = 4.1$	
7	(i) Given that $(y = Ae^{px})$, where A and p are constants, find an expression for $(\frac{dy}{dx})$. Solution: Differentiate y with respect to x : $\frac{dy}{dx} = \frac{d}{dx}(Ae^{px}) = A \times \frac{d}{dx}(e^{px})$ Recall that $(\frac{d}{dx}(e^{px}) = pe^{px})$: $\frac{dy}{dx} = A \times pe^{px}$ So, $\frac{dy}{dx} = Ape^{px}$ (ii) Hence, find the value of A and of p for which $(\frac{dy}{dx} + 4y = 14e^{3x})$. Solution: Substitute $(y = Ae^{px})$ and $(\frac{dy}{dx} = Ape^{px})$ into the equation: $Ape^{px} + 4Ae^{px} = 14e^{3x}$ Factor (Ae^{px}): $Ae^{px}(p + 4) = 14e^{3x}$ For this equation to hold for all x, the exponents and coefficients must match: 1. Exponent matching: $(px = 3x \rightarrow p = 3)$ 2. Coefficient matching: $(A(p + 4) = 14)$ Substitute p = 3: $A(3 + 4) = 14 \rightarrow A \times 7 = 14 \rightarrow A = 2$ Final Answers:	

	$A = 2, \quad p = 3$	
8	Given: $(N = 10\,000e^{-0.2t})$ (i) Find the initial population of the insects. The initial population is when $(t = 0)$: $N = 10\,000 \times e^{-0.2 \times 0} = 10\,000 \times e^0 = 10\,000 \times 1 = 10\,000$ Answer: The initial population is 10,000 insects. (ii) Estimate the number of insects after 5 days. Substitute $t = 5$: $N = 10\,000 \times e^{-0.2 \times 5} = 10\,000 \times e^{-1}$ Now, $(e^{-1} \approx 0.3679)$: $N \approx 10\,000 \times 0.3679 = 3,679$ Answer: After 5 days, there are approximately 3,679 insects. (iii) When will the colony first decrease to 1,000 insects? Set $(N = 1,000)$ and solve for t: $\frac{1,000}{10,000} = e^{-0.2t}$ $0.1 = e^{-0.2t}$ Take the natural logarithm of both sides: $\ln(0.1) = -0.2t$ $t = \frac{\ln(0.1)}{-0.2}$ Now, $(\ln(0.1) \approx -2.3026)$: $t = \frac{-2.3026}{-0.2} = 11.513$ Answer: The colony will first decrease to 1,000 insects after approximately 11.5 days.	