

4 ESSENTIAL QUESTIONS TO MASTER

TRIGONOMETRY

1	The function f is defined, for all values of x by $f(x) = 2\cos\left(\frac{x}{2}\right) + c,$ where (c) is a constant. The graph of $(y = f(x))$ passes through the point $\left(\frac{2\pi}{3}, 2\right)$ (i) Find the value of (c). (ii) State the amplitude and period of (f).	
2	A, B and C are the angles of a triangle. (i) Show that $\tan C = -\tan(A + B).$ Angle $(A = 45^\circ)$ and angle $(B = 60^\circ)$. (ii) Without using a calculator, find $(\tan C)$ in the form $(a + \sqrt{b})$, where a and b are integers.	
3	Do not use a calculator in this question. (i) Show that $\frac{\sin(a + \beta) + \sin(a - \beta)}{\cos(a + \beta) + \cos(a - \beta)} = \tan a.$ Angles A and B are both acute and $(\sin(A - B) = -\frac{3}{5})$. (ii) What can be deduced about the size of B relative to A? (iii) Find the value of $\cos(A - B)$.	
4	(a) Solve the equation $(1 + \tan x) \sec x = 6 \operatorname{cosec} x$ for $0^\circ \leq x \leq 360^\circ$ (b) Prove that $(\sec x - \tan x)(\csc x + 1) = \cot x$	

Answer Key:

1	(i) We have $f\left(\frac{2\pi}{3}\right)$	
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	$= 2\cos\left(\frac{1}{2} \times \frac{2\pi}{3}\right) + c$ $= 2\cos\left(\frac{\pi}{3}\right) + c$ $= 2 \times \frac{1}{2} + c$ $= 1 + c.$ Since $f\left(\frac{2\pi}{3}\right) = 2,$ $1 + c = 2$ $\Rightarrow$ $c = 1.$ (ii) The function is $f(x) = 2\cos\left(\frac{x}{2}\right) + 1.$ - Amplitude = 2 (the coefficient of the cosine). - Angular frequency ($b = \frac{1}{2}$) so the period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}}$ $= 4\pi$
2	(i) Since $(A + B + C = 180^\circ)$, we have $C = 180^\circ - (A + B).$ But $(\tan(180^\circ - x) = -\tan x)$. Hence $\tan C = \tan(180^\circ - (A + B))$ $= -\tan(A + B),$ as required. (ii) We are given $(A = 45^\circ), (B = 60^\circ)$, so $A + B = 105^\circ,$ $C = 180^\circ - 105^\circ = 75^\circ.$ Thus we need $(\tan 75^\circ)$ Use the angle-sum formula with $(75^\circ = 45^\circ + 30^\circ)$:

	$\tan(45^\circ + 30^\circ)$ $= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{-1 \times \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ Rationalize the denominator: $\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ $= \frac{(\sqrt{3}+1)^2}{3-1}$ $= \frac{3+2\sqrt{3}+1}{2}$ $= \frac{4+2\sqrt{3}}{2}$ $= 2 + \sqrt{3}$ Therefore $\tan C = \tan 75^\circ = 2 + \sqrt{3}$
3	(i) Proof of the identity Use the sum-and-difference formulas: $\sin(a + \beta)$ $= \sin a \cos \beta + \cos a \sin \beta,$ $\sin(a - \beta)$ $= \sin a \cos \beta - \cos a \sin \beta.$ Adding, $\sin(a + \beta) + \sin(a - \beta)$ $= 2 \sin a \cos \beta.$ Similarly, $\cos(a + \beta)$ $= \cos a \cos \beta - \sin a \sin \beta$

	$\cos(a - \beta)$ $= \cos a \cos \beta + \sin a \sin \beta,$ so $\cos(a + \beta) + \cos(a - \beta) = 2 \cos a \cos \beta.$ Therefore $\frac{\sin(a + \beta) + \sin(a - \beta)}{\cos(a + \beta) + \cos(a - \beta)} = \frac{2 \sin a \cos \beta}{2 \cos a \cos \beta}$ $= \frac{\sin a}{\cos a}$ $= \tan a$ as required. (ii) Sign of $(\sin(A - B))$ Since A and B are acute, $(A, B \in (0, 90^\circ))$ so their difference A-B lies in $(-90^\circ, 90^\circ)$. On that interval $(\sin \theta)$ is negative exactly when $(\theta < 0)$. We are given $\sin(A - B) = -\frac{3}{5} < 0,$ so $A - B < 0$. Hence $A < B.$ (iii) Value of $(\cos(A - B))$ We know $\sin(A - B) = -\frac{3}{5}.$ Then $\cos^2(A - B)$
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	$= 1 - \sin^2(A - B)$ $= 1 - \left(-\frac{3}{5}\right)^2$ $1 - \frac{9}{25} = \frac{16}{25}$ so $\cos(A - B) = \pm \frac{4}{5}.$ But since $(A - B \in (-90^\circ, 90^\circ))$, its cosine is positive. Therefore $\boxed{\cos(A - B) = \frac{4}{5}}.$
4	(a) $(1 + \tan x) \sec x = 6 \operatorname{cosec} x$ $\frac{1}{\cos x} (1 + \tan x) = \frac{6}{\sin x}$ $\tan x (1 + \tan x) = 6$ $\tan^2 x + \tan x - 6 = 0$ $\tan x = 2 \quad \text{or} \quad \tan x = -3$ $x = 63.435^\circ, 243.435^\circ \quad \text{or} \quad x = 108.435^\circ, 288.435^\circ$ $x = 63.4^\circ, 243.4^\circ \quad \text{or} \quad x = 108.4^\circ, 288.4^\circ$ (b) 1. Start with the left-hand side: $\text{LHS} = ((\sec x - \tan x)(\csc x + 1)).$ 2. Write everything in terms of sine and cosine: $= \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \left(\frac{1}{\sin x} + 1 \right)$ $= \frac{1 - \sin x}{\cos x} \times \frac{1 + \sin x}{\sin x}.$ 3. Multiply the numerators and the denominators: $= \frac{(1 - \sin x)(1 + \sin x)}{\sin x \cos x}$ $= \frac{1 - \sin^2 x}{\sin x \cos x}.$

4. Use the Pythagorean identity ($1 - \sin^2 x = \cos^2 x$):

$$\begin{aligned}
 &= \frac{\cos^2 x}{\sin x \cos x} \\
 &= \frac{\cos x}{\sin x} \\
 &= \cot x.
 \end{aligned}$$

Thus LHS ($= \cot x$), which is the RHS. Q.E.D.