

8 ESSENTIAL QUESTIONS TO MASTER

SURDS

1	Express $((7 + \sqrt{5})^2 + (4 - \sqrt{5})^2)$ in the form $(a + b\sqrt{5})$, stating the value of each of the integers (a) and (b) .	
2	It is given that $(p = \sqrt{5} - \sqrt{3})$ (i) Without using a calculator, express (p^2) in the form $(\sqrt{a} - 2\sqrt{b})$, where (a) and (b) are integers. (ii) Hence, or otherwise, find the exact value of $3p^4 - 8p^2 + 1$.	
3	Without using a calculator, express $(\frac{11-\sqrt{7}}{3+\sqrt{7}})$ in the form $(a + b\sqrt{7})$, where (a) and (b) are integers.	
4	Given that $x = \frac{\sqrt{17}-4}{\sqrt{17}+4}$, show that $x + \frac{1}{x} = -6$.	
5	Given that $\left(\frac{3}{\sqrt{2}} - \frac{\sqrt{6}}{3} + \frac{30}{\sqrt{600}}\right) \times \frac{12}{\sqrt{3}-1}$ $= a\sqrt{2} + b\sqrt{6}$ find a and b .	
6	Express $\frac{(5\sqrt{2}-3)(7+\sqrt{2})}{3+\sqrt{2}}$ in the form $(p\sqrt{2} + q)$, where p and q are rational numbers.	
7	(a) Simplify $(3\sqrt{180} + \sqrt{245} - 2\sqrt{125})$, leaving your answer in surd form. (b) Given that $\sqrt{a + b\sqrt{3}} = \frac{13}{4+\sqrt{3}}$, where a and b are integers, find a and b .	
8	Solve the equation $2x^3 - 3x^2 - x + 1 = 0$. If a root is not exact, express it in the form $\frac{(c \pm \sqrt{d})}{2}$ where c and d are integers.	

Answer Key:

1	1. Expand each square: $(7 + \sqrt{5})^2 = 7^2 + 2 \times 7 \times \sqrt{5} + (\sqrt{5})^2$ $= 49 + 14\sqrt{5} + 5$ $= 54 + 14\sqrt{5}$ $(4 - \sqrt{5})^2 = 4^2 - 2 \times 4 \times \sqrt{5} + (\sqrt{5})^2$ $= 16 - 8\sqrt{5} + 5$ $= 21 - 8\sqrt{5}$ 2. Add them: $(54 + 14\sqrt{5}) + (21 - 8\sqrt{5})$ $= (54 + 21) + (14\sqrt{5} - 8\sqrt{5})$ $= 75 + 6\sqrt{5}$ Hence in the form $(a + b\sqrt{5})$ we have $(a = 75),$ $(b = 6)$
2	(i) Compute $p^2 = (\sqrt{5} - \sqrt{3})^2$ $= 5 + 3 - 2\sqrt{15}$ $= 8 - 2\sqrt{15}.$ To put this in the form $(\sqrt{a} - 2\sqrt{b})$, note $(8 = \sqrt{64})$. Hence $p^2 = \sqrt{64} - 2\sqrt{15},$ so $(a = 64)$ and $(b = 15).$ (ii) Let $(x = p^2 = 8 - 2\sqrt{15}).$ Then $p^4 = x^2 = (8 - 2\sqrt{15})^2$ $= 64 + 4 \times 15 - 32\sqrt{15}$ $= 124 - 32\sqrt{15}$ Hence $3p^4 - 8p^2 + 1$ $= 3(124 - 32\sqrt{15}) - 8(8 - 2\sqrt{15}) + 1$ $= (372 - 96\sqrt{15}) - (64 - 16\sqrt{15}) + 1$ Combine like terms: $372 - 64 + 1 = 309 \quad -96\sqrt{15} + 16\sqrt{15} = -80\sqrt{15}.$

	Therefore $3p^4 - 8p^2 + 1 = 309 - 80\sqrt{15}.$
3	We want to simplify $\frac{11 - \sqrt{7}}{3 + \sqrt{7}}$ into the form $(a + b\sqrt{7})$. 1. Multiply top and bottom by the conjugate of the denominator, $(3 - \sqrt{7})$: $\frac{11 - \sqrt{7}}{3 + \sqrt{7}} \times \frac{3 - \sqrt{7}}{3 - \sqrt{7}} = \frac{(11 - \sqrt{7})(3 - \sqrt{7})}{(3 + \sqrt{7})(3 - \sqrt{7})}.$ 2. Expand numerator and denominator: Numerator: $\begin{aligned} &(11 - \sqrt{7})(3 - \sqrt{7}) \\ &= 11 \times 3 - 11\sqrt{7} - 3\sqrt{7} + (\sqrt{7})^2 \\ &= 33 - 14\sqrt{7} + 7 \\ &= 40 - 14\sqrt{7}. \end{aligned}$ Denominator (difference of squares): $\begin{aligned} &(3 + \sqrt{7})(3 - \sqrt{7}) \\ &= 3^2 - (\sqrt{7})^2 \\ &= 9 - 7 \\ &= 2. \end{aligned}$ 3. Divide through by 2: $\frac{40 - 14\sqrt{7}}{2} = 20 - 7\sqrt{7}.$

	So in the form $(a + b\sqrt{7})$, we have $\boxed{20 - 7\sqrt{7}},$ $a = 20$ and $b = -7$.
4	1. Write down (x) and its reciprocal: $x = \frac{\sqrt{17}-4}{\sqrt{17}+4},$ $\frac{1}{x} = \frac{\sqrt{17}+4}{\sqrt{17}-4}.$ 2. Form the sum: $x + \frac{1}{x} = \frac{\sqrt{17}-4}{\sqrt{17}+4} + \frac{\sqrt{17}+4}{\sqrt{17}-4}.$ 3. Combine over the common denominator $((\sqrt{17}+4)(\sqrt{17}-4))$: $x + \frac{1}{x} = \frac{(\sqrt{17}-4)^2 + (\sqrt{17}+4)^2}{(\sqrt{17}+4)(\sqrt{17}-4)}$ 4. Simplify numerator and denominator separately: - Denominator: $ \begin{aligned} &(\sqrt{17}+4)(\sqrt{17}-4) \\ &= (\sqrt{17})^2 - 4^2 \\ &= 17 - 16 \\ &= 1. \end{aligned} $ - Numerator: $ \begin{aligned} &(\sqrt{17}-4)^2 + (\sqrt{17}+4)^2 \\ &= (17 - 8\sqrt{17} + 16) \\ &\quad + (17 + 8\sqrt{17} + 16) \end{aligned} $

	$= (17 + 16 + 17 + 16) + (-8\sqrt{17} + 8\sqrt{17})$ $= 66 + 0$ $= 66$ 5. Therefore $x + \frac{1}{x}$ $= \frac{66}{66}$ $= \frac{1}{1}$ $= 66$ as required.
5	1. Simplify each term inside the first parenthesis. $\frac{30}{\sqrt{600}}$ $= \frac{30}{10\sqrt{6}}$ $= \frac{3}{\sqrt{6}}$ $= \frac{3\sqrt{6}}{6}$ $= \frac{\sqrt{6}}{2}$ So $\frac{3}{\sqrt{2}} - \frac{\sqrt{6}}{3} + \frac{30}{\sqrt{600}}$ $= \frac{3}{\sqrt{2}} - \frac{\sqrt{6}}{3} + \frac{\sqrt{6}}{2}$ 2. Combine the $(\sqrt{6})$- terms: $-\frac{1}{3} + \frac{1}{2} = -\frac{2}{6} + \frac{3}{6} = \frac{1}{6},$ so $\frac{3}{\sqrt{2}} - \frac{\sqrt{6}}{3} + \frac{\sqrt{6}}{2}$ $= \frac{3}{\sqrt{2}} + \frac{1}{6}\sqrt{6}$ $= \frac{3\sqrt{2}}{2} + \frac{\sqrt{6}}{6}$

	3. Multiply by $\frac{12}{\sqrt{3}-1}$: $\frac{\frac{\sqrt[3]{2}}{2} + \frac{\sqrt{6}}{6}}{12 \times \left(3\sqrt{\frac{2}{2}} + \sqrt{\frac{6}{6}}\right)}$ $= \frac{\sqrt{3}-1}{18\sqrt{2}+2\sqrt{6}}$ 4. Rationalize the denominator $((\sqrt{3}-1))$ by multiplying top and bottom by $((\sqrt{3}+1))$: $\frac{18\sqrt{2}+2\sqrt{6}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ $= \frac{(18\sqrt{2}+2\sqrt{6})(\sqrt{3}+1)}{3-1}$ $= \frac{(24\sqrt{2}+20\sqrt{6})}{2}$ $= 12\sqrt{2}+10\sqrt{6}$ Hence $a\sqrt{2}+b\sqrt{6}=12\sqrt{2}+10\sqrt{6}$ $so(a=12, b=10.)$
6	Solution: 1. Expand the numerator: $(5\sqrt{2}-3)(7+\sqrt{2})$ $= 5\sqrt{2} \times 7 + 5\sqrt{2} \times \sqrt{2} - 3 \times 7 - 3\sqrt{2}$ $= 35\sqrt{2} + 10 - 21 - 3\sqrt{2}$ $= 32\sqrt{2} - 11$ 2. So the expression becomes $\frac{32\sqrt{2}-11}{3+\sqrt{2}}$ 3. Rationalize the denominator by multiplying top and bottom by $(3-\sqrt{2})$: $\frac{32\sqrt{2}-11}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} =$

	$\frac{(32\sqrt{2} - 11)(3 - \sqrt{2})}{3^2 - (\sqrt{2})^2}$ $\frac{(96\sqrt{2} - 64 - 33 + 11\sqrt{2})}{9 - 2}$ $\frac{107\sqrt{2} - 97}{7}$ 4. Finally, split into the form $(p\sqrt{2} + q)$: $\frac{107\sqrt{2} - 97}{7}$ $= \frac{107}{7}\sqrt{2} - \frac{97}{7}$ Hence $\frac{(5\sqrt{2} - 3)(7 + \sqrt{2})}{3 + \sqrt{2}}$ $= \frac{107}{7}\sqrt{2} - \frac{97}{7}$ so $(p = \frac{107}{7})$ and $(q = -\frac{97}{7})$
7	(a) $3\sqrt{180} + \sqrt{245} - 2\sqrt{125}$ $= 3\sqrt{36 \times 5} + \sqrt{49 \times 5} - 2\sqrt{25 \times 5}$ $= 3 \times 6\sqrt{5} + 7\sqrt{5} - 2 \times 5\sqrt{5}$ $= 18\sqrt{5} + 7\sqrt{5} - 10\sqrt{5}$ $= (18 + 7 - 10)\sqrt{5}$ $= 15\sqrt{5}$ (b) First rationalize the right-hand side: $\frac{13}{4 + \sqrt{3}} \times \frac{4 - \sqrt{3}}{4 - \sqrt{3}}$ $= \frac{13(4 - \sqrt{3})}{4^2 - (\sqrt{3})^2}$

	$= \frac{13(4 - \sqrt{3})}{16 - 3}$ $= \frac{13(4 - \sqrt{3})}{13}$ $= 4 - \sqrt{3}$ Hence $\sqrt{a + b\sqrt{3}} = 4 - \sqrt{3}.$ Square both sides: $a + b\sqrt{3} = (4 - \sqrt{3})^2$ $= 16 - 8\sqrt{3} + 3$ $= 19 - 8\sqrt{3}$ By comparing coefficients, $a = 19, \quad b = -8.$
8	1. Original equation $2x^3 - 3x^2 - x + 1 = 0.$ 2. Test for a simple (rational) root By the Rational-Root Test, any rational root must be among $(\pm 1, \pm \frac{1}{2})$. Try $(x = \frac{1}{2})$: $2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1$ $= 2 \times \frac{1}{8} - 3 \times \frac{1}{4} - \frac{1}{2} + 1$ $= \frac{1}{4} - \frac{3}{4} - \frac{1}{2} + 1 = 0$ Hence $x = \frac{1}{2}$ is a root. 3. Factor out $(x - \frac{1}{2})$

Divide $(2x^3 - 3x^2 - x + 1)$ by $x - \frac{1}{2}$. One finds

$$\begin{aligned}
 &2x^3 - 3x^2 - x + 1 \\
 &= \left(x - \frac{1}{2}\right)(2x^2 - 2x - 2).
 \end{aligned}$$

4. Solve the remaining quadratic
Set

$$\begin{aligned}
 2x^2 - 2x - 2 &= 0 \\
 \Rightarrow \\
 x^2 - x - 1 &= 0.
 \end{aligned}$$

Apply the quadratic formula $\left(x = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}\right)$ with $(a = 1)$, $(b = -1)$, $(c = -1)$:

$$\begin{aligned}
 x &= \frac{1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\
 &= \frac{1 \pm \sqrt{1 + 4}}{2} \\
 &= \frac{1 \pm \sqrt{5}}{2}
 \end{aligned}$$

5. List all solutions
Therefore the three roots are

$$x = \frac{1}{2}, \quad x = \frac{1 + \sqrt{5}}{2}, \quad x = \frac{1 - \sqrt{5}}{2}.$$