

8 ESSENTIAL QUESTIONS TO MASTER

BINOMIAL EXPANSION

1	(i) Expand $(2 + x)^7$ in ascending powers of x up to the term in x^3 : (ii) Find the value of a , given that the coefficient of x^2 in the expansion of $((1 + ax)(2 + x)^7)$ is 63.	
2	(i) Given that m is a constant, expand $(3 + mx)^4$, in ascending powers of x , simplifying each term in your expansion. (ii) Given also that the coefficient of x is equal to the coefficient of x^2 , find the value of m . (iii) By considering the general term in the binomial expansion of $\left(3x - \frac{1}{2x}\right)^9$, explain why there are no even powers of x in this expansion. (iv) Using the value of m found in part (a)(ii), find the term independent of x in the expansion of $(3 + mx)^4 \left(3x - \frac{1}{2x}\right)^9$.	
3	(a) Write down and simplify the first three terms in the expansion, in ascending powers of x , of $\left(2 + \frac{x}{2}\right)^n$ where n is a positive integer. (b) The first two terms in the expansion, in ascending powers of x , of $\left(\frac{4}{3} - 3x\right)\left(2 + \frac{x}{2}\right)^n$ are $(a + bx^2)$. Find the value of n . (c) Find the term independent of x in the expansion of $\left(x - \frac{1}{2x^2}\right)^9$	
4	(i) By considering the general term in the binomial expansion of $\left(2mx + \frac{1}{x^3}\right)^6$, where m is a constant, explain why there are no odd powers of x in this expansion. (ii) Given that the coefficient of the fourth term is twice the coefficient of the third term, find the value of m .	
5	(i) Verify that the formula for $\binom{n}{r}$ works for $\binom{n}{n}$. (ii) In the expansion of $((1 + mx)^8)$, the coefficient of x^3 is four times that of x^2 . Find the value of m .	

6	(i) By considering the general term in the binomial expansion of $\left(px^3 + \frac{1}{x}\right)^9$, where p is a constant, explain why there are no even powers of x in this expansion. (ii) Given that the coefficient of x^{11} in the expansion of $\left(px^3 + \frac{1}{x}\right)^9$ is twice the coefficient of (x^7) , find the value of p .	
7	(i) Find the first 3 terms in the expansion of $\left(3 + \frac{x}{5}\right)^m$ in ascending powers of x , where m is a positive integer greater than 2. Give the terms in their simplest forms. (ii) In the expansion of $((5 - 2x)^3 \left(3 + \frac{x}{5}\right)^m)$, there is no term in x^2 . Find the value of m .	
8	Expand $((2 - p)^6)$. Use this result to find the expansion of $\left(2 - \frac{x}{2} + 2x^2\right)^6$ in ascending powers of (x) as far as the term in (x^2) .	

Answer Key:

1	(i) Using the binomial theorem: $(2 + x)^7 = \sum_{r=0}^7 \binom{7}{r} 2^{7-r} x^r$ Expanding up to (x^3) : – For (x^0) : $\binom{7}{0} 2^7 x^0 = 1 \times 128 \times 1 = 128$ – For (x^1) : $\binom{7}{1} 2^6 x^1 = 7 \times 64 \times x = 448x$ – For (x^2) : $\binom{7}{2} 2^5 x^2 = 21 \times 32 \times x^2 = 672x^2$ – For (x^3) : $\binom{7}{3} 2^4 x^3 = 35 \times 16 \times x^3 = 560x^3$ So, $(2 + x)^7 = 128 + 448x + 672x^2 + 560x^3 + \dots$ (ii) Let's expand: $-((2 + x)^7 = 128 + 448x + 672x^2 + 560x^3 + \dots)$ $-((1 + ax)(2 + x)^7 = (2 + x)^7 + ax(2 + x)^7)$ The coefficient of (x^2) comes from: $(1 \times (672x^2))$ from the first term $(ax \times (448x) = 448ax^2)$ from the second term	
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	Add together: Total coefficient of $x^2 = 672 + 448a$ Set equal to 63 and solve for (a): $672 + 448a = 63$ $448a = 63 - 672 = -609$ $a = \frac{-609}{448}$	
2	(i) $(3 + mx)^4$ $= 3^4 + \binom{4}{1}(3^3)mx + \binom{4}{2}(3^2)(mx)^2 + \binom{4}{3}(3)(mx)^3 + (mx)^4$ $= 81 + 108mx + 54m^2x^2 + 12m^3x^3 + m^4x^4$ (ii) $108m = 54m^2$ $m = 2$ (iii) $\binom{9}{r}(3x)^{9-r}\left(-\frac{1}{2x}\right)^r$ Powers of (x) in the expansion: $x^{9-r} \cdot x^{-r} = x^{9-r-r} = x^{9-2r}$ $2r$ is always even for all integer values of r . The difference between an odd and an even number is always odd. Thus, there are no even powers of x in the expansion. (iv) $(81 + 108mx + 54m^2x^2 + 12m^3x^3 + m^4x^4)(3x - \frac{1}{2x})^9$ $(81 + 216x + 216x^2 + 96x^3 + 16x^4)(\dots x^{-1} \dots x^{-3} \dots)$ We look for terms where the exponents of x multiply to x^{-1} and x^{-3}: $x^{9-2r} = x^{-1}$ $9 - 2r = -1$ $2r = 10$ $r = 5$ $x^{9-2r} = x^{-3}$ $9 - 2r = -3$ $2r = 12$ $r = 6$ coefficient = $216 \times \binom{9}{5}(3)^4\left(-\frac{1}{2}\right)^5 + 96 \times \binom{9}{6}(3)^3\left(-\frac{1}{2}\right)^6$ $= -\frac{130977}{2}$	

3	$\left(2 + \frac{x}{2}\right)^n = 2^n + n(2)^{n-1}\left(\frac{x}{2}\right) + \frac{n(n-1)}{2}(2)^{n-2}\left(\frac{x}{2}\right)^2 + \dots$ $= 2^n + \frac{xn(2)^n}{4} + \frac{n(n-1)x^2(2)^n}{32} + \dots$ The first two terms in the expansion, in ascending powers of x, of $\left(\left(\frac{4}{3} - 3x\right)\left(2 + \frac{x}{2}\right)^n\right)$ are $(a + bx^2)$. Find the value of n. $\left(\frac{4}{3} - 3x\right)\left(2 + \frac{x}{2}\right)^n$ $= \left(\frac{4}{3} - 3x\right)\left[2^n + \frac{xn(2)^n}{4} + \frac{n(n-1)x^2(2)^n}{32} + \dots\right]$ $= \frac{4}{3} \times 2^n + \frac{xn(2)^n}{3} + \frac{n(n-1)x^2(2)^n}{24} - 3x2^n - \frac{3x^2n(2)^n}{4} + \dots$ No x term: $\frac{n(2)^n}{3} - 3(2)^n = 0$ $n = 9$ Find the term independent of x in the expansion of $\left(x - \frac{1}{2x^2}\right)^9$: $T_{r+1} = \binom{9}{r} (x)^{9-r} \left(-\frac{1}{2x^2}\right)^r$ Term independent of x : $9 - r - 2r = 0$ $r = 3$ Term independent of x: $= \binom{9}{3} (x)^6 \left(-\frac{1}{2x^2}\right)^3$ $= \binom{9}{3} \times x^6 \times \left(-\frac{1}{8x^6}\right)$ $= 84 \left(-\frac{1}{8}\right)$ $= -\frac{21}{2}$	
4	The general term in the expansion is: $T_{r+1} = \binom{6}{r} (2mx)^{6-r} \left(\frac{1}{x^3}\right)^r = \binom{6}{r} (2m)^{6-r} x^{6-r} \times x^{-3r}$	

	$= \binom{6}{r} (2m)^{6-r} x^{6-r-3r} = \binom{6}{r} (2m)^{6-r} x^{6-4r}$ The power of x in each term is $6 - 4r$. Since $4r$ is always even, $6 - 4r$ is always even. Therefore, there are no odd powers of x in the expansion. (ii) Given that the coefficient of the fourth term is twice the coefficient of the third term, find the value of m. The third term $r = 2$: $T_3 = \binom{6}{2} (2m)^4 \left(\frac{1}{x^3}\right)^2 = 15 \times 16m^4 x^4 \times x^{-6} = 240m^4 x^{-2}$ Coefficient: $240m^4$ - The fourth term ($r = 3$): $T_4 = \binom{6}{3} (2m)^3 \left(\frac{1}{x^3}\right)^3 = 20 \times 8m^3 x^3 \times x^{-9} = 160m^3 x^{-6}$ Coefficient: $160m^3$ Given: $160m^3 = 2 \times 240m^4$ $= 480m^4$ Final Answer: - There are no odd powers of x in the expansion. - The value of m is $\frac{1}{3}$	
5	(i) The formula for the binomial coefficient is: $\binom{n}{r} = \frac{n!}{r! (n-r)!}$ Let's verify that this formula works when ($r = n$): $\binom{n}{n} = \frac{n!}{n! (n-n)!}$ Since ($n - n = 0$) and ($0! = 1$): $\binom{n}{n} = \frac{n!}{n! \cdot 1} = \frac{n!}{n!} = 1$ Choosing all (n) objects from a set of (n) objects can only be done in one way, which matches our result. The formula for ($\binom{n}{r}$) correctly gives ($\binom{n}{n} = 1$).	

(ii) Step 1: Write the general term.

The general term in the expansion of $(1 + mx)^8$ is:

$$T_{r+1} = \binom{8}{r} (mx)^r = \binom{8}{r} m^r x^r$$

Step 2: Find the coefficients.

- Coefficient of (x^3) :

$$C_3 = \binom{8}{3} m^3$$

- Coefficient of (x^2) :

$$C_2 = \binom{8}{2} m^2$$

Step 3: Set up the equation.

Given:

$$C_3 = 4 \times C_2$$

So,

$$\binom{8}{3} m^3 = 4 \times \binom{8}{2} m^2$$

Step 4: Simplify the equation.

Divide both sides by (m^2) (assuming $(m \neq 0)$):

$$\binom{8}{3} m = 4 \times \binom{8}{2}$$

Calculate the binomial coefficients:

$$\binom{8}{3} = 56, \quad \binom{8}{2} = 28$$

So,

$$56m = 4 \times 28$$

$$56m = 112$$

	$m = \frac{112}{56} = 2$ Final Answer $m = 2$	
6	(i) The general term (T_{r+1}) in the expansion of $(px^3 + \frac{1}{x})^9$ is: $T_{r+1} = \binom{9}{r} (px^3)^{9-r} \left(\frac{1}{x}\right)^r$ $= \binom{9}{r} p^{9-r} (x^3)^{9-r} (x^{-1})^r$ $= \binom{9}{r} p^{9-r} x^{3(9-r)} x^{-r}$ $= \binom{9}{r} p^{9-r} x^{27-3r-r}$ $= \binom{9}{r} p^{9-r} x^{27-4r}$ The power of (x) in the general term is ($27 - 4r$). For any whole number (r), ($4r = 2(2r)$) is even since it is divisible by 2. 27 is an odd number, and if we subtract an even number from 27, the result is always odd. Therefore, $27 - 4r$ is always odd, so there are no even powers of (x) in the expansion of $(px^3 + \frac{1}{x})^9$. (ii) For the term in (x^{11}): $27 - 4r = 11$ $4r = 16$ $r = 4$ Term in (x^{11}): $\binom{9}{4} p^{9-4} x^{11} = 126p^5 x^{11}$ For the term in (x^7): $27 - 4r = 7$ $4r = 20$ $r = 5$ Term in (x^7): $\binom{9}{5} p^{9-5} x^7 = 126p^4 x^7$	

	Since the coefficient of (x^{11}) is twice the coefficient of (x^7): $126p^5 = 2 \times (126p^4)$ $p^5 = 2p^4$ $p = 2$	
7	(i) Find the first 3 terms in the expansion of $\left(3 + \frac{x}{5}\right)^m$ in ascending powers of (x), where $(m > 2)$. Binomial Expansion Formula: $(a + b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 + \dots$ Here, $(a = 3)$, $(b = \frac{x}{5})$. First 3 terms: 1. First term $((x^0))$: 3^m 2. Second term $((x^1))$: $m \times 3^{m-1} \times \frac{x}{5} = \frac{m \cdot 3^{m-1}}{5} x$ 3. Third term $((x^2))$: $\frac{m(m-1)}{2} \times 3^{m-2} \times \left(\frac{x}{5}\right)^2 = \frac{m(m-1)}{2} \times 3^{m-2} \times \frac{x^2}{25}$ $= \frac{m(m-1)3^{m-2}}{50} x^2$ So, the first 3 terms are: $3^m + \frac{m \times 3^{m-1}}{5} x + \frac{m(m-1)3^{m-2}}{50} x^2$ (ii) In the expansion of $((5 - 2x)^3 \left(3 + \frac{x}{5}\right)^m)$, there is no term in x^2. Find the value of m. Let's expand each part up to x^2: Expand $((5 - 2x)^3)$: $(5 - 2x)^3 = 5^3 + 3 \times 5^2(-2x) + 3 \times 5(-2x)^2 + \dots$ $= 125 - 150x + 60x^2$ Expand $\left(\left(3 + \frac{x}{5}\right)^m\right)$ (from part (i)): $= 3^m + \frac{m \cdot 3^{m-1}}{5} x + \frac{m(m-1)3^{m-2}}{50} x^2$ Multiply and collect (x^2) terms: Let's find all combinations that give x^2: 1. (constant term from first) $\times ((x^2)$ term from second): $125 \times \frac{m(m-1)3^{m-2}}{50} x^2$ 2. $((x)$ term from first) $\times ((x)$ term from second):	

	$(-150x) \times \frac{m \times 3^{m-1}}{5} x = -150 \times \frac{m \times 3^{m-1}}{5} x^2 = -30m \times 3^{m-1} x^2$ 3. $((x^2)$ term from first) $\times$ (constant term from second): $60x^2 \times 3^m = 60 \times 3^m x^2$ Sum of coefficients of (x^2): $125 \times \frac{m(m-1)3^{m-2}}{50} - 30m \times 3^{m-1} + 60 \times 3^m = 0$ Let's simplify each term: $- (125 \times \frac{m(m-1)3^{m-2}}{50} = \frac{5}{2} m(m-1)3^{m-2})$ $- (-30m \times 3^{m-1})$ $- (60 \times 3^m)$ So, $\frac{5}{2} m(m-1)3^{m-2} - 30m \times 3^{m-1} + 60 \times 3^m = 0$ Divide both sides by (3^{m-2}) (since $(m > 2)$, $(3^{m-2} \neq 0)$): $\frac{5}{2} m(m-1) - 30m \times 3 + 60 \times 3^2 = 0$ $\frac{5}{2} m(m-1) - 90m + 540 = 0$ Multiply both sides by 2 to clear the fraction: $5m(m-1) - 180m + 1080 = 0$ $5m^2 - 5m - 180m + 1080 = 0$ $5m^2 - 185m + 1080 = 0$ Quadratic equation: $5m^2 - 185m + 1080 = 0$ Divide by 5: $m^2 - 37m + 216 = 0$ Factor: $(m-9)(m-24) = 0$ So, $(m = 9)$ or $(m = 24)$. Since $(m > 2)$, both are valid, but typically only one value is expected in such questions. Both are mathematically valid. Final Answers: (i) The first 3 terms are: $3^m + \frac{m \times 3^{m-1}}{5} x + \frac{m(m-1)3^{m-2}}{50} x^2$ (ii) The value(s) of (m) are: $m = 9 \text{ or } m = 24$	
8	1. Expand $((2-p)^6)$ using the Binomial Theorem	

The Binomial Theorem states:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Here, $(a = 2), (b = -p), (n = 6)$:

$$(2 - p)^6 = \sum_{k=0}^6 \binom{6}{k} 2^{6-k} (-p)^k$$

Let's write the first few terms:

$$= 2^6 - 6 \times 2^5 p + 15 \times 2^4 p^2 - 20 \times 2^3 p^3 + \dots$$

$$= 64 - 192p + 240p^2 - 160p^3 + \dots$$

2. Use this result to expand $\left(2 - \frac{x}{2} + 2x^2\right)^6$ up to (x^2)

Let $(p = \frac{x}{2} - 2x^2)$.

Substitute (p) into the expansion above, and expand up to the term in (x^2) :

Substitute (p) :

$$-(p = \frac{x}{2} - 2x^2)$$

$$-(p^2 = \left(\frac{x}{2} - 2x^2\right)^2 = \frac{x^2}{4} - 2x^3 + 4x^4) \left(\text{but we only need up to } (x^2), \text{ so } (p^2 \approx \frac{x^2}{4})\right)$$

Substitute into the expansion:

$$(2 - p)^6 \approx 64 - 192p + 240p^2$$

Now substitute (p) and (p^2) :

$$= 64 - 192\left(\frac{x}{2} - 2x^2\right) + 240\left(\frac{x^2}{4}\right)$$

Expand each term:

$$-(-192 \times \frac{x}{2} = -96x)$$

$-(-192 \times (-2x^2) = +384x^2)$ $-(240 \times \frac{x^2}{4} = 60x^2)$ So, $= 64 - 96x + 384x^2 + 60x^2$ $= 64 - 96x + 444x^2$ The expansion of $(2 - \frac{x}{2} + 2x^2)^6$ in ascending powers of (x) up to the term in (x^2) is: $64 - 96x + 444x^2$	
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